AM 289 N1

THE OPEN UNIVERSITY

Arts/Mathematics
An Interfaculty Second Level Course
History of Mathematics
Counting, Numerals and Calculation 1

Counting I: Primitive and More Developed Counting Systems

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3 *ukasar-urapon*
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6 *ukasar-ukasar-ukasar*

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COUNTING I: PRIMITIVE AND MORE DEVELOPED COUNTING SYSTEMS

Prepared by B. L. van der Waerden and Graham Flegg for the Course Team

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1.1

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General Introduction to Units N1–N5

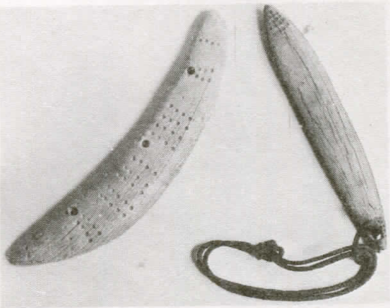
Number plays an essential part in our culture as indeed it does in any recognizable form of society. However far we go back in history, we can be certain that number has played its part in man's ways of thought and in his reaction to the world in which he has found himself. The special area of study *Counting, Numerals and Calculation* is essentially about number in some of its many aspects.

Above: Marriage tablet from the Kai Islands (Linden Museum, Stuttgart)

Right: Irish gravestone from Aglish with Ogham characters (National Museum of Ireland).

In these five units we shall not be concerning ourselves with the abstract mathematical properties associated with number[1], but with the practical problems of counting, writing numbers and performing simple arithmetical operations. We shall begin by considering the counting systems which exist or have existed at various times and places. Counting systems are directly associated with the spoken word and sometimes with gestures of parts of the body. We shall discuss the number words of languages both familiar and unfamiliar and this will in turn involve us in discussion of relationships between languages, particularly those of the Indo-European family to which English belongs. We shall consider in some

1 These are considered particularly in Units 2 and 10 of the 'general' section of the course.

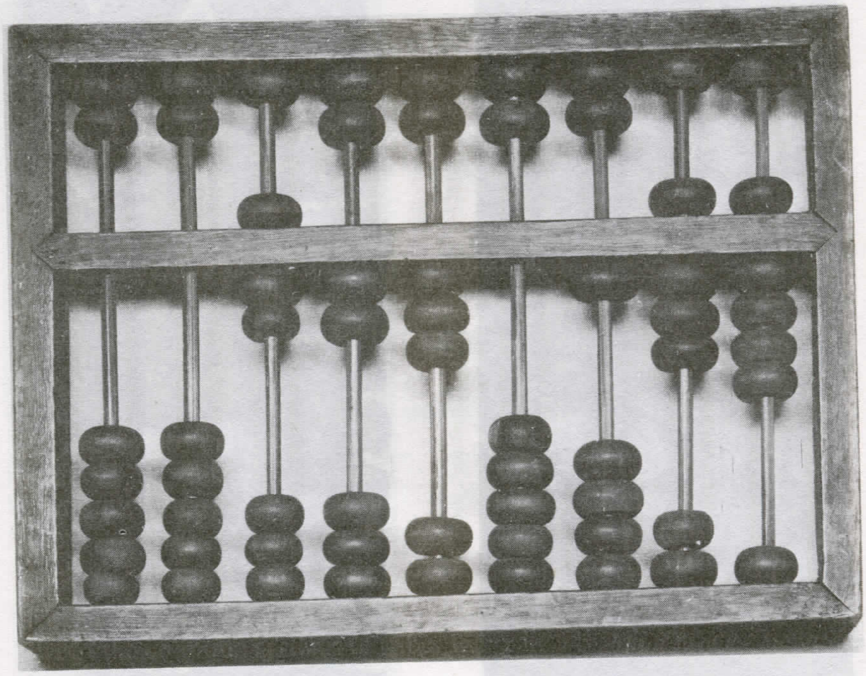


Turkana cattle tallies (Merseyside County Museums, Liverpool).

Peruvian quipu (Linden Museum, Stuttgart).

detail the history of our own familiar decimal system of counting and we shall discuss rudimentary methods of recording numbers (tally-sticks and knots) before continuing with a survey of the various ways of writing numerals in ancient times and more recently in many parts of the world. Here again we shall be especially concerned with our own decimal system for writing numbers (including fractions), its origin and its history to the present day.

Number has many practical everyday applications which involve arithmetical calculation. In our familiar decimal system such calculation is comparatively easy. With other systems for writing numbers it has often not been so easy and early recourse was made to various aids to computation, the most common of which was the counting board or abacus. The history of such aids will be discussed in the last of these units.



Chinese abacus (Crown copyright, Science Museum, London).

The examination of the various aspects of number which we are able to undertake in these five units can make no claim to be exhaustive. Essentially we can do little more than lay the basis for a serious study of the cultural aspects of number. From time to time we shall quote extracts from papers and books which treat the subject matter at greater depth than we can attempt in the units. Those of you who are stimulated to pursue the subject beyond the confines of this course are strongly recommended to study such works in their entirety. In particular, we would refer you to Karl Menninger's cultural history of numbers *Zahlwort und Ziffer* (1958) in its English translation by Paul Broneer (1969): *Number Words and Number Symbols*, MIT Press.

Introduction to *Counting I and II* (Units N1 and N2)

In the first two units of this special area of study we shall be concerned with counting systems and number words. In addition to our own familiar decimal counting system, we shall consider several other counting methods which were and (in many cases) are still used in parts of Africa, Asia, Australia and the Americas. Such methods include those that we call *2-count*, *4-count*, *5–10-count*, *5–20-count*, and *counting on fingers and toes*. We shall also include discussion of the Babylonian *10–60-system*. We

shall, however, restrict our discussion to the more important and characteristic systems. We shall begin with the most primitive counting methods and we shall end with decimal number words and with tally-sticks and knots used to indicate numbers.

The history of counting systems and number words is one of the most interesting parts of the history of civilization. It is interesting not only in itself but also because of its close connection with the history of mathematics and the history of science in general. A good counting system, which enables us to express large numbers in spoken number words or written numerals, is an essential prerequisite for arithmetic, algebra, astronomy, physics and many other studies.

As far as possible we shall try to investigate not only the present state of various counting systems but also their history. We can trace our own decimal counting system and number words over a period of nearly five thousand years. We can do this because the English language is akin to more ancient Germanic languages, such as Gothic, which in turn belongs to the same great language family as Latin, Greek, Old Persian and Sanskrit. The languages of this family are called *Indo-European languages*, because the area in which they were and still are spoken extends from India to Europe. All these languages are supposed to be derived from one hypothetical 'Original Indo-European language', which already existed in the period between 3000 and 2500 BC. In this original language number words up to 'hundred' (or perhaps 'thousand') existed and the method of forming composite number words like 'eight hundred and forty-five' was just the same as in our own language. All this is explained in Unit N2.

However, when we turn to the counting systems of primitive tribes, we know only the recent state. In exceptional cases, such as the Mayan and Aztec systems, we can go back one or two millennia but not more. If we want to obtain information about the history of primitive counting methods, such as *counting by pairs*, we have to make hypotheses and these are always uncertain.

Extent of the Aztec kingdom in c. 1486.

Extent of Aztec influence up to the arrival of the Spaniards in 1519.

The area of the Mayan and Aztec civilizations.

Diffusion versus Independent Invention

When we find similarities between the counting systems of tribes living in different countries or even in different continents, there are always two possible assumptions that we can make: we can assume a *diffusion* hypothesis (i.e. spreading from one country or continent to another) or an *independent invention* hypothesis (i.e. separate inventions of the same principle in many different places). In certain cases the diffusion hypothesis can be proved. For example when indigenous tribes in South America are found to be using Spanish number words, or when the coloured population of the United States is found to be speaking English and using the decimal counting system, we can establish clear cases of diffusion.

However, in other cases where there is no direct historical evidence to support the diffusion hypothesis, the majority of scholars working in this field tend to assume independent invention. There is one notable exception to this rule, A. Seidenberg.[1] In a highly interesting paper 'The Diffusion of Counting Practices' (*University of California Publications in Mathematics*, 3, 4, 1960), he puts forward the hypothesis that every one of the known counting systems was invented just once in one geographical location and then spread from this place of invention to those places where we find it in use today.

Professor A. Seidenberg.

How should we approach the opposing hypotheses of diffusion and independent invention? Should we seek to be entirely convinced by one of them, or should we take up a position somewhere between the two? Of

1 Professor Seidenberg discusses the *diffusion* hypothesis in the radio programme accompanying Unit N1.

course we cannot ask you to make up your mind at this point because you have not yet seen the arguments for and against each hypothesis or studied the evidence in particular cases. However as you study the units you may well find yourself thinking that we are presenting only those arguments which favour the diffusion hypothesis and are therefore doing insufficient justice to the alternative hypothesis of independent invention which, at first sight, may seem the more plausible of the two. In fact, the situation is that it is extremely difficult to find soundly based factually supported arguments in favour of independent invention. When assuming independent invention, scholars just present the argument that 'diffusion has not been conclusively proved', even though it is clear that this in itself cannot possibly justify the independent invention hypothesis. In all those cases where both hypotheses are possible, there seems to be an almost automatic preference for independent invention. The reasons for this are by no means clear.

The discovery of a good counting system, which enables a man to count better and more easily than his father did, means a great advance in civilization. Such discoveries are very rare: there are still tribes who cannot count beyond 5 or 6. Also, better counting systems have a tendency to spread and to replace more primitive methods.

For these reasons, therefore, we have shown a preference for the diffusion hypothesis in the units. If two counting systems are found to be very similar in many details, the diffusion hypothesis immediately explains why they are similar, whereas the hypothesis of independent invention supplies no direct explanation at all. In what follows our main purpose is to present the relevant facts. We shall also explain certain hypotheses (chiefly those of Seidenberg) and discuss the supporting arguments but it is important to remember that other hypotheses, in particular that of independent invention, are also possible.

Unit N1 Counting I: Primitive and More Developed Counting Systems

Objectives

After studying this unit and listening to the associated radio programme *Diffusion of Counting*, you should be able to:

1 Explain the distinction between *systemless counting* and a *counting system.*

2 Describe and discuss the various counting systems.

3 Distinguish between the *independent invention* and the *diffusion* hypotheses, and discuss evidence for and against each hypothesis.

4 Discuss evidence from number words and number signs which relates to the various counting systems.

5 Describe the *binary system* of numbers and distinguish it from *2-count*.

6 Discuss the difficulties of distinguishing between related systems such as *5–10-* and *5–20-count*.

7 Sketch and interpret staircase diagrams representing counting systems.

8 Describe the principal geographical locations of the various counting systems and interpret their distribution.

9 Discuss evidence relating to the origins of the quinary, vigesimal, sexagesimal and other systems.

10 Describe and discuss the significance of *finger* and *body counting*.

11 Comment briefly on difficulties associated with discussion of the origin of counting.

A Note on Exercises

A number of exercises have been included in the text. While some of these are specifically designed for self-assessment purposes, i.e. to ensure that you have understood the salient points in the preceding sections, others are there to encourage you to summarize or interpret facts which have been presented and to draw conclusions of your own. Unless otherwise stated your answers should be restricted to two or three short sentences.

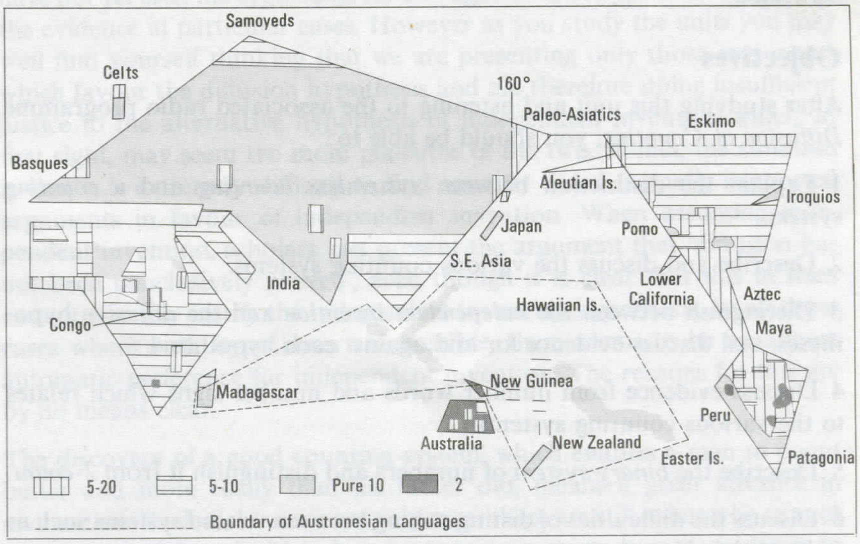
You are strongly recommended to attempt each exercise as it appears in the text before reading further. A specimen answer (SA) has been provided, where appropriate, immediately following each exercise. Naturally it is in your own interest to look at an SA *only after you have completed your attempt at answering the question concerned.*

Radio

The radio programme accompanying this unit is entitled *Diffusion of Counting*. In this programme Professor A. Seidenberg of the University of California and Graham Flegg (Course Team Chairman) describe a number of counting systems and their geographical distribution and discuss evidence in favour of the diffusion hypothesis (see pp. 7–8).

During the programme reference is made to the *2-system* (counting by pairs) and, in particular, to the three examples which are tabled on p. 15 (the Gumulgal, the Bakairi and the Bushmen). You will also be referred to Map 1 (p. 16), 'Pure 2- and neo-2-systems', to the sketch map below, 'Counting systems of the world', and to Persian numerals (illustrated on p.19), Egyptian numerals (on p. 22) and Sumerian numerals (on p. 21).

You may also find it useful to be able to refer to Maps 2 (p. 39) and 3 (p. 42).



Sketch map of counting systems of the world (by kind permission of Professor A. Seidenberg). This map is based on Father W. Schmidt's map XIII in his Sprachfamilien, Atlas. *Schmidt's original map is reliable except for the 5–10-system markings. It is roughly correct if these markings are taken to mean that words for 6, 7, 8 and 9 are of composite form, i.e. formed as 5 + 1, 5 + 2, etc.*

For those who may wish to follow up the references in the programmes to works of Conant, McGee, *et al.*, the details are:

Conant, L. L. (1923) *The Number Concept*, Macmillan.

McGee, W. J. (1900) *Primitive Numbers*, 19th Annual Report, Bureau of American Ethnology.

Schmidt, W. (1926) *Sprachfamilien und Sprachenkreise der Erde*, Heidelberg.

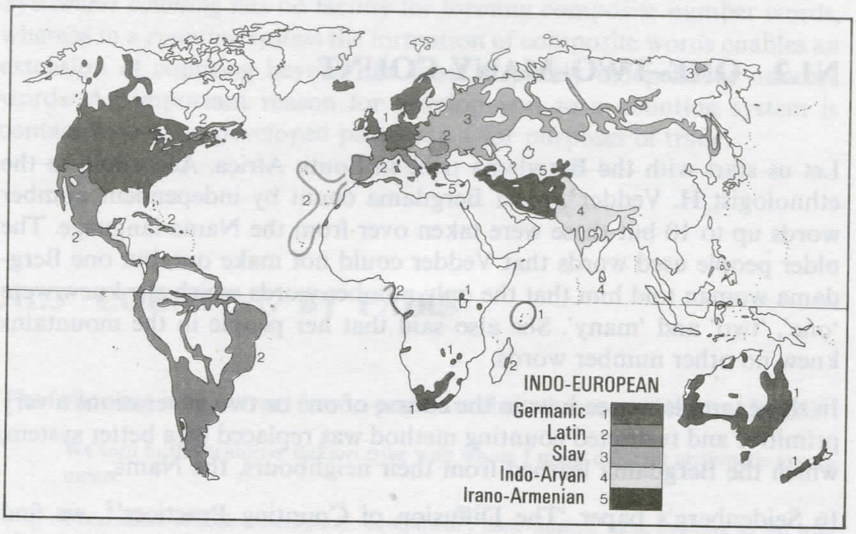
Von den Steinen, K. (1894) *Unter den Naturvölkern Zentral Brasiliens*, Berlin.

N1.1 THE DECIMAL SYSTEM

Our counting system is based on the number ten. Why just ten? Probably because we have ten fingers. Finger counting is widespread among both primitive and civilized peoples.

The main principle of the decimal system is that ten is considered as a new unit, from which point counting starts again. The multiples of ten are counted by the same system: ten, twenty, thirty, etc. Ten tens are again a new unit and so on.

This way of counting is very old. English is a Germanic language and the oldest known Germanic language, namely Gothic, the language of the Bible translation of the bishop Wulfila (*c.* AD 350), already has the same method of counting. Essentially the same system is also found in all Romance languages, such as French and Italian, which are derived from Latin. The Greek language, which existed as early as 1200 BC, has the same counting method and so has Hittite, which was being spoken in Asia Minor by 1800 BC. All these languages belong to the 'Indo-European' family of languages. As 'Indo-European' indicates, these languages were (and still are) spoken in a region ranging from India to Europe. They are supposed to stem from one mother language, the 'original Indo-European language'. This original mother language must have been in existence by between 3000 and 2500 BC. Since all Indo-European languages have the same counting system and very similar number words (compare e.g. our 'two' and 'three' with Latin *duo* and *tres*), we may safely conclude that the Indo-European mother language had the same method of counting.



The distribution of Indo-European languages today.

Let us call this method the *decimal counting system.* It should be clearly distinguished from the *decimal number notation.* Spoken words like 'twenty-three' or 'three-and-twenty' belong to the decimal counting system, whereas written numerals like '23' belong to the decimal number notation, which is comparatively recent, as we shall see later in Unit N3.

Exercise 1

(a) Why can we conclude that the decimal counting system is at least as old as the 'original Indo-European language'?

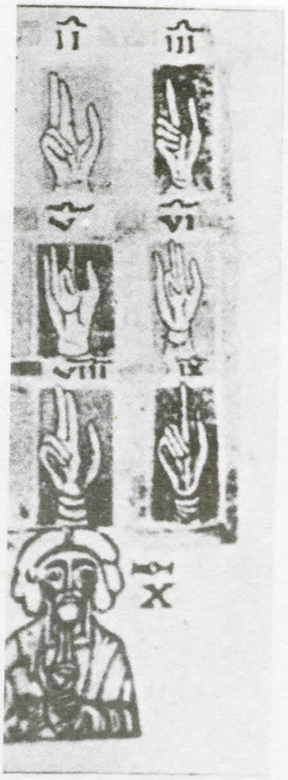
(b) Indicate which of the following belong to the decimal counting system:

(i) 723

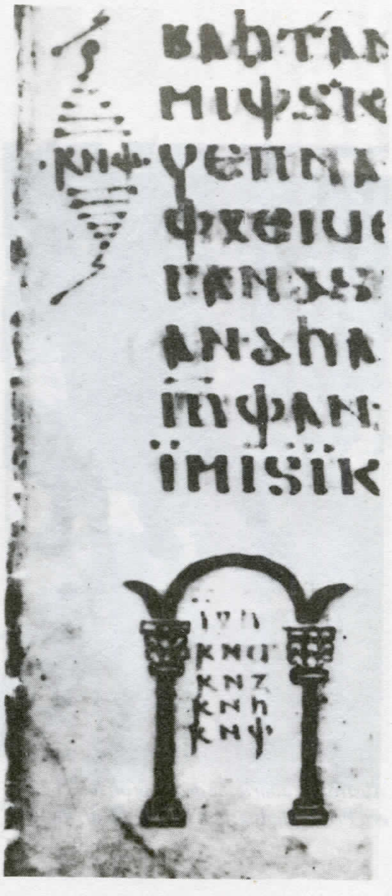
(ii) 'seven-and-twenty'

(iii) 'two-thousand-and-thirty-three'

(iv) 'three-dozen'



Finger numbers from a thirteenth-century Spanish codex, *in Luca Paccioli's* Summa de arithmetica, geometria proportioni et proportionalita, *1494 (British Museum).*



Left-hand margin of folio 113 r of the Gothic Bible, showing verse 159 and references to the verses in the arch at the bottom (Universitetsbiblioteket, Uppsala).

SA 1

(a) Since all Indo-European languages have the same (decimal) counting system and similar number words, we can conclude that the 'mother language' had a decimal counting system.
(b) (ii) and (iii) only. (i) belongs to the decimal number notation and (iv) includes the non-decimal word 'dozen'.

In this unit and in Unit N2, we shall be concerned with counting systems only and not with systems of numerals. Numerals will be considered in Unit N3.

The origins of our decimal counting system are hidden in obscurity because Indo-European languages were spoken long before they were written. What we can observe up to the present is the spread or diffusion of our counting system from more civilized to less civilized people. Whoever wants to trade with us has to learn our money system, which is in turn closely connected with our method of counting. The black population of North America speaks English and uses the decimal system, whereas their ancestors in Africa had more primitive number systems, as we shall see presently. Quite generally, more efficient counting systems tend to spread and more primitive methods tend to disappear. (We shall elaborate on this later.)

We shall now continue by looking at some of the other counting methods differing from our own and used by ancient civilized peoples and by primitive tribes, and we shall compare these with the decimal counting system to see if there are special ways in which they are related.

N1.2 ONE–TWO–MANY-COUNT

Let us start with the Bergdama tribe in South Africa. According to the ethnologist H. Vedder[1], most Bergdama count by independent number words up to 10 but these were taken over from the Nama language. The older people used words that Vedder could not make out, but one Bergdama woman told him that the only number words which she knew were 'one', 'two' and 'many'. She also said that her people in the mountains knew no other number words.

In this example we see how in the course of one or two generations a very primitive and restricted counting method was replaced by a better system, which the Bergdama learned from their neighbours, the Nama.

In Seidenberg's paper 'The Diffusion of Counting Practices'[2], we find the following paragraph:

> Counting may seem to be an elementary process that any human being would come across by himself, yet the fact that many groups of savages cannot count beyond 2 makes us realize that the process is an advanced one. Over most of aboriginal Australia one finds essentially only two number words, 'one' and 'two': many of the tribes are reported definitely as not counting beyond 2, and as indicating higher multiplicities by the word 'many', while others go a little further by compounding these words—for example, expressing 3 as 'two-one', 4 as 'two-two', and 5 as 'two-two-one'. A similar method of counting, the so-called 2-system, is found in New Guinea, in South America, and in South Africa.

Australian aborigine drawing on sand (Camera Press, photograph Richard Harrington).

The fact that many Australian aboriginal tribes, for example, cannot count beyond 2 is indicative of a comparatively primitive state of civilization. Contact with a more advanced state soon demands that such a limited

1 Vedder, H. (1923) *Die Bergdama*, Hamburg, p. 165.
2 *Op. cit.*, p. 216.

counting system be extended. It may be that these peoples have been able to count on their fingers without using words, but as soon as it becomes necessary to trade with goods or money the need for more number words becomes imperative.

A somewhat higher level of counting is reported of the Kalahari Bushmen in Africa. S. S. Dorman writes[1]:

> They have no words beyond five, and not often beyond three, for they cannot count any farther. I have satisfied myself on that point by putting some Kalahari Bushmen to the test on more than one occasion.

Counting up to 3 or 5 with no facility for any further detailed counting cannot be considered to be a counting system. If in any part of the world we find only a set of independent number words like 'one, two, three' or 'one, two, three, four, five', we shall call this *systemless counting*. On the other hand, if we find a method to form composite number words like 'fifteen' or 'fifty', we shall use the expression 'counting system'.

We now proceed to discuss counting systems, starting with the most primitive of all.

Exercise 2

Briefly explain the difference between *systemless counting* and a *counting system*, and give one important cause of the development in a primitive tribe from one to the other.

SA 2

Systemless counting has no facility for forming composite number words, whereas in a *counting system* the formation of composite words enables an extension of counting beyond the limit of purely independent number words. An important reason for development to a counting system is contact with more developed peoples, e.g. for purposes of trade.

N1.3 COUNTING BY PAIRS

The following is an extract from a paper on the tribesmen of central Brazil[2]:

> We turn to the primitive Bakairi tribe with whom I made different arithmetic experiments.
>
> The names of their calculation 'equipment', their fingers, have nothing to do with numbers. The thumb is called 'father', and the little finger is called 'child' or 'little one'. The middle finger is, as with us, the 'middle'; indexfinger and ringfinger become 'neighbours'—that is, at least, the obvious interpretation—of the 'father' and the 'little one'.
>
> They count in words up to 6:
>
> 1 = *tokále* 4 = *ahage aháge*
> 2 = *aháge* 5 = *aháge aháge tokále*
> 3 = *aháge tokále* or *ahewáo* 6 = *aháge aháge aháge*
>
> It is noticeable that the numbers 1 to 3 have their own names, and that the names for the numbers 4 to 6 are composites of *aháge* and *tokále*. I became acquainted with the word *ahewáo* for 3 on my second expedition, but it is used as often as the composite 'two-one'. The idea that *ahewáo* might be a new word is wrong; comparing the languages proves that the older Karaibi tribes, north of the Amazon, had previously used

Brazilian tribeswoman (Musée de l'Homme, Paris).

1 Dorman, S. S. (1925) *Pygmies and Bushmen of the Kalahari*, London, p. 61.

2 Von den Steinen, K. (1894) *Unter den Naturvölkern Zentral Brasiliens*, Berlin, pp. 406–7, translated by Margo McIntosh.

ahewáo. It is noticeable that 3 (*ahewáo*) is used in none of the higher figures, not even in the number word for 6.

When we remove the independent word *ahewáo*, the Baikairi counts:

one	two-two
two	two-two-one
two-one	two-two-two

Beginning with the little finger of the left hand he says *tokále*, then he joins fingers IV and V and says *ahâge*. He keeps finger III separated from IV and V and says *ahâge tokále*; he joins fingers II and III and says *ahâge ahâge*, touches the thumb and says *ahâge ahâge tokále*, and pushes the little finger of the right hand against the fingers of the left hand and says *ahâge ahâge ahâge*. After 6 the Baikairi is not able to count further; he just touches fingers IV, III, II and I of the right hand and repeats *méra* ('this one'), then he touches the toes of his left foot and after that the toes of his right foot, and with every touch he says *méra*. If he wants to express more than 20, he tousles his hair in all directions saying *méra méra*.

Logically, they should call 7 *ahâge ahâge ahâge tokále*, but they would have to count the number words themselves and they cannot do this. Even when it affects important matters, their lack of words and their inability even to indicate higher amounts than 20 is an invincible obstacle. Obviously after 6 they have only a vague approach to higher amounts. Imagine that somebody is climbing a staircase and for some reason intends to count the steps, but suddenly realizes that he does not know for certain how many steps he has climbed so far. Disconcerted, he gets more and more mixed up and so has no idea how many steps he has climbed. The Baikairi climbs the staircase knowing that after the toes of his first foot, he climbed more steps than the fingers of his left hand, after the toes of the second foot he climbed more steps than after the toes of his first foot. That 10 fingers and 5 toes represent 15 is beyond their imagination.

Their conception after 6 is only clear to the extent that they can touch fingers and toes. As soon as precise amounts have to be counted their ruses hardly reach further than their knowledge of figures. I have tried experiments in the Paleko and Tumayaua districts—I could tousle my own hair when I think about it—but will mention just the example of a large amount of corn seeds. When I put ten corn seeds in front of them and asked *âtúra* ('how many'), they counted them in the way described above, slowly but correct up to 6. For the seventh and eighth corn seed they showed fingers IV and V of the right hand and said *méra méra*, but after that they were exhausted. They reminded me of people who are playing cards without interest and soon yawn and say 'I have no talent for playing cards'. The Baikairis yawned as well and when I urged them to carry on they pulled faces, moaned about *kinaráchu zwáno* which means 'brain work' but can also mean 'headache', and sometimes they just walked out. In any event, they went on strike.

Exercise 3

(a) Make your own summary of the evidence recorded in the extract above.

(b) What is particularly noteworthy about the Baikairis' words for numbers up to 6 together with their associated finger gestures?

SA 3

(b) The composite words for the numbers 3 and above made up of the words for 1 and 2, together with associated gestures which involve a pairing of fingers.

From this evidence we see that the Bakairis are effectively obliged to pair off objects to be counted. When asked to count or give the number of as few as three grains of corn, it is recorded that they separate them into a pile of two and the remaining one. In one of their folk-tales, in order to express the fact that the hero felled five trees, they use three sentences: 'He felled two trees. Again he felled two trees. He felled another.'

Let us call this method *counting by pairs*. It is closely connected with a set of number words formed by repeating the words for 1 and 2. We have already seen another example in the extract from Seidenberg's paper (p. 12). We shall discuss this system of counting by looking at three examples taken from the same paper.

	Gumulgal (Australia)	Bakairi (South America)	Bushman (South Africa)
1	*urapon*	*tokale*	*xa*
2	*ukasar*	*ahage*	*t'oa*
3	*ukasar-urapon*	*ahage tokale* (or *ahewao*)	*'quo*
4	*ukasar-ukasar*	*ahage ahage*	*t'oa-t'oa*
5	*ukasar-ukasar-urapon*	*ahage ahage tokale*	*t'oa-t'oa-ta*
6	*ukasar-ukasar-ukasar*	*ahage ahage ahage*	*t'oa-t'oa-t'oa*

In Africa, *pure 2-count* is found only among the Bushmen. They count by this method up to $10 = 2 + 2 + 2 + 2 + 2$. Beyond 10 the method does not work: the number words become too long. To see this, consider how a listener, having only a pure 2-count, would interpret 'two and two and two and two and two and two and two and two'. He would need a more sophisticated counting system in order to count how often the word 'two' has been said.

In Australia, pure 2-count is used by several tribes in the east, and it is also found in nearby New Guinea (see Map 1, p. 16, black circles). This region is small and connected and all tribes in it speak closely related languages. It is hardly a reasonable hypothesis that pure 2-count was independently invented five times in one region having the same language group throughout, so we may reasonably suppose that the method was invented only once and then spread over the eastern region of Australia and New Guinea.

Among the South American Indians we find pure 2-count in a connected region in the north-east. Here too it seems reasonable to suggest that it was invented only once in South America. In the paper from which we have already quoted Seidenberg explains the striking uniformity in the formation of number words in Africa, Australia and South America by making an even bolder guess.

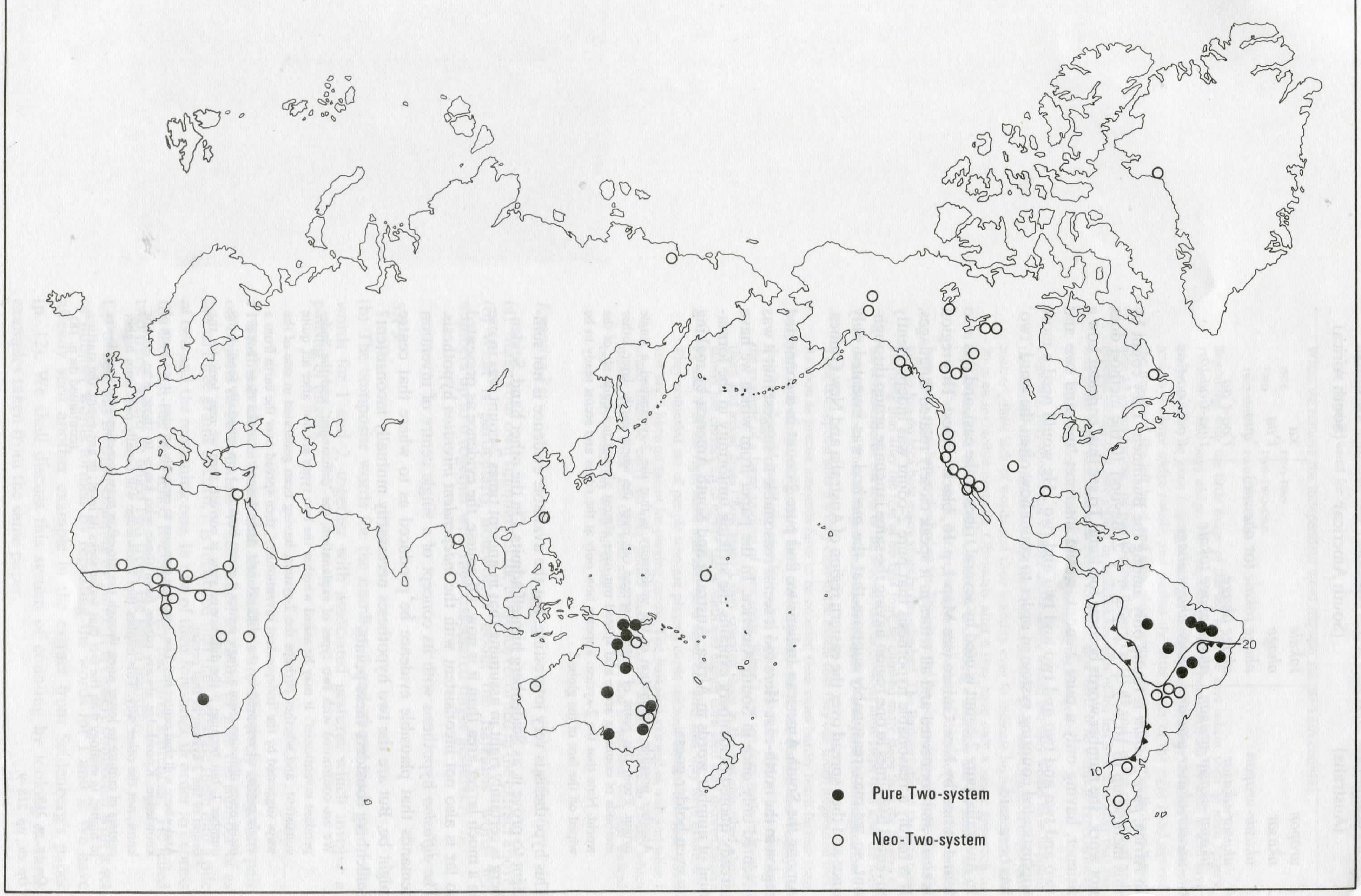
> Another possible explanation of the uniformity is that the 2-system had a single origin. From this point of origin it spread out over the whole earth; later, other methods of counting arose and spread out over almost all, but not quite all, of the world. Note that the 2-system appears now only at the edges and seems ready to be wiped off the face of the globe.[1]

This hypothesis is very interesting, but the available evidence is not sufficient to prove it, as Seidenberg himself admits. On the other hand, Seidenberg is certainly right in assuming that in ancient times 2-count was in use in a much larger area than it is now. Of course, the evidence as presented so far is also not inconsistent with the *independent invention* hypothesis. The *diffusion* hypothesis with its concept of a single centre of invention demands that plausible evidence be produced as to where that centre might be. But are the two hypotheses necessarily mutually inconsistent? Seidenberg considers these points.[2]

> We are confronted with two types of explanation, the 'diffusionist' and the 'independent inventionist'. It may be asked whether the two types are not, after all, quite consistent; and whether perhaps the 2-system, having been generated in one of the ways suggested by the 'independent inventionists', then spread over the earth from a single center, as required by the 'diffusionists'. Here I need to make clear that if I insist upon diffusion from a *single* center it is not because I am especially fond of the number 1, but because if the diffusionist view is correct, then we may look for the origin of the 2-system in that center.
>
> And where will that center be? The 2-system appears to embody an idea; it is a form of knowledge. Knowledge always passes from those who know to those who do not know, not the other way. This implies that knowledge will usually pass from higher centers of culture to lower ones, though the direction may sometimes be from lower to higher, in analogy with the fact that water seeks its level; in any event, the starting

(continued on p. 18).

1 *Op. cit.*, p. 218.
2 *Op. cit.*, pp. 218–9.



MAP 1 Pure 2- and neo-2-counting (by kind permission of Professor A Seidenberg). (A key to this map appears opposite.)

Key to Map 1

Pure 2-Counting

(with at least $2+2$, $2+2+1$)

South America

Left to right:

Zamuco	$2+1, 2+2, \ldots, 2+2+2+2+1$
Arara	$2+1, 2+2, 2+2+1$
Bakairi	$2+1, 2+2, 2+2+1, 2+2+2$
Cayapo	$2+1, 2+2, 2+2+1, 2+2+2$
Crengez	$2+2, 2+2+1$
Apinage	$2+1, 2+2, 2+2+1$
Krao	$2+1, 2+2, 2+2+1, 2+2+2$
Canella	$2+1, 2+2, 2+2+1, 2+2+2$
Cotoxo	$2+1, 2+2, 2+2+1$

Australia

Bottom to top:

South Narrinyeri	$2+2, 2+2+1, 2+2+2, 2+2+2+1$, etc.
North Kuri	$2+1, 2+2, 2+2+1$
Minyung	$2+1, 2+2, 2+2+1, 2+2+2$, etc.
Kana	$2+1, 2+2, 2+2+1$
(Gualluma	$2+2, 2+2+1$; see Neo-2-Counting)
Kohoyimidir	$2+1, 2+2, 2+2+1$
Kauralgal	$2+1, 2+2, 2+2+1, 2+2+2, 2+2+2+1, 2+2+2+2$
Gumulgal	$2+1, 2+2, 2+2+1, 2+2+2$
Torres Straits	$2+1, 2+2, 2+2+1, 2+2+2$

New Guinea

Parb	$2+2, 2+2+1$
Sisiami	$2+1, 2+2, 2+2+1$
Anal; Arop	$2+1, 2+2, 2+2+1$

South Africa

Bushman	$2+2, 2+2+1, \ldots, 2+2+2+2+2$

Neo-2-Counting

South America

Bottom to top:

Tsonega	$2\times3, 2\times4, 2\times5$
Tehuelche	$2\times3, 2\times4$
Guarani	$3\times2, 4\times2, 2\times6, 6+7, 2\times9, 2\times9+1, 4\times5, 2\times4\times5$
Toba	$2+1, 2+3, 2\times3, 1+2\times3, 2\times4, 2\times4+1$
Caingua	$3\times2+1, 2\times4$
Payagua	$2\times3, 2\times4, 2\times4+1, 2\times5$
Cherente	$3\times2, 4\times2$

Note: The fainter, but still fairly clear, trace 2×3 is found in the region between the pure 10-counters and 5–20-counters amongst the following:

Tiatinagua	2×3
Zapara	$2\times2, 2\times2+1, 2\times3, 2\times3+1$ (to 7 only)

Also with the:

Ona	$2\times2, 2\times3$
Marawan	2×3 (6 contains 3)

North America

West Coast, bottom to top:

Yuma	$3\times2, 4\times2$
Paiute	$3\times2, 4\times2$
Tobikhan	$3\times2, 2\times4, 2\times5$
Netela	$3\times2, 4\times2$
Salinas	$3\times2, 8-1, 4\times2, 10-1$
Wintun	$2\times3, 6+1, 2\times4, 8+1, 2\times5$
Yana	$3\times2, 4\times2$
Tehama	$2\times3, 2\times4$
Haida	$2\times2, 3\times2, 4\times2$
Kygani	$2\times2, 3\times2, 4\times2$

Interior, left to right:

Tukudh	$2+1, 2\times3, 2\times4$
Kutchin	$2\times3, 2\times4$
Tacully	$2\times3, 2\times4$
Dogrib	$2\times3, 2\times4$
Sarcis	$2\times3, 2\times4$
Sicanni	$2\times3, 2\times4$
Beaver	$2\times3, 2\times4$
Montagnais	$2\times3, 2\times4$
Chippewa	$2\times3, 2\times4$
Digger (Huerfano Park)	$2\times3, 2\times4$
Atakapa (on the Gulf)	$3\times2, 4\times2$.

Eskimo:

Labrador	$3\times2, 4\times2, 5\times2$
Melville Bay, Greenland	$3\times2, 3+4, 4\times2, 4+5, 5\times2$

Asia

Yukaghir	$2\times3, 2\times4$
Tsuihoan (Formosa)	$2\times3, 2\times4$
Karen (South Burma)	$3\times2, 3\times2+1, 4\times2+1$

Nicobar Islands:

Inland tribes	3×2
Coast tribes	2×4

Pacific

Radak Group, Marshall Islands	$2\times3, 2\times3+1, 2\times4+1$

New Guinea:

Waima, Roro, Kabadi, Pokau	$2\times3, 2\times3+1, 2\times4, 2\times4-1$
Hula, Keapara, Galoma	$2\times3, 2\times4-1, 2\times4, 10-1$

Australia, right to left:

Kamilaroi	$2+2, 2+3, 3+3$
Wirri-Wirri	$2+1, 2+2, 3+2, 3+3$
Gualluma (Upper Sherlock River)	$2+2, 2+2+1, 3+3$

Africa

Sudan, left to right:

Gurunsi	$6+1, 8+1$
Munsi	$3+3, 4+4$
Ekio, Akparabon, Ododop, *et alia*	$3+3, 4+3, 4+4, 5+4$
Sao Group (Kotoko of Kousri, of Logone, of Goulfei; Logone)	$2\times3, 2\times4$
Mbai, Sara Gulei	$3+3, 4+3, 4+4, 4+5$

Boa Group:

Niellim	$2\times3, 2\times4$
Tounia	$3\times2, 4+4$
Ndonga	$2\times3, 2\times4$
Mongwandi	$3+3, 4+3$
Kanderma	$2\times3, 4+3$

Bantu, top to bottom:

Bali (Cameroons)	$6+1, 8+1$
Kombe	$6+1, 8+1$
Luba-Hemba	$8-1, 10-1$
Konde	7, 8 bracketed; 9, 10 bracketed

point will be a center of knowledge. It is known that ideas arose in the areas of the ancient civilizations; it is not known that ideas ever arose, anciently, anywhere else. Therefore, we will look in the ancient centers of civilization for the origin of the 2-system.

. . . .

The theory of a single origin of counting explains the uniformity without explicitly explaining the 2; the theory of independent invention, on the other hand, is obliged immediately to ponder the 2, and soon comes to the suggestion that we psychoanalyze ourselves to find out what the savage really knows. The diffusionist theory, for its part, having indicated historical reasons for the uniformity, is committed to a historical research.

Note The 'ways suggested by the independent inventionists' (line 4 of the above extract) include the 'duality' concept (by which we distinguish another person from ourselves), the 'two-sided' concept (by which we distinguish what we *can* see 'in front of us' from what we *cannot* see 'behind us'), and the 'pair-wise' concept (by which we recognize in eyes, hands, feet, etc., the idea of 'left and right').

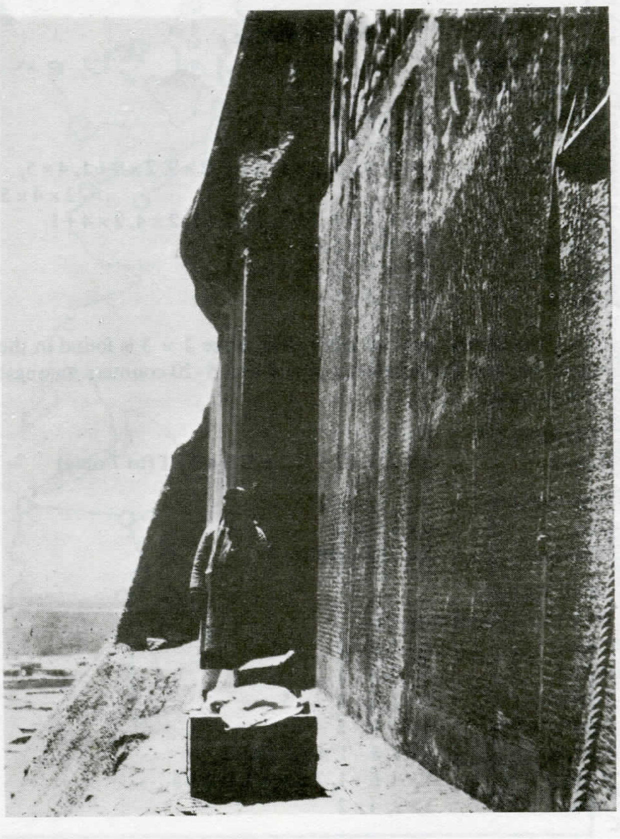
Exercise 4

On what basis does Seidenberg associate the independent invention hypothesis with psychological research and the diffusion hypothesis with historical research?

SA 4

Independent invention is 'obliged immediately to ponder the 2' and thus has to invent psychological hypotheses for spontaneous generation of 2-systems in many different locations. *Diffusion* has to explain historically the geographical distribution of 2-count ('without explicitly explaining the 2') and to produce historical evidence for an origin in one ancient centre of civilization.

In support of his hypothesis Seidenberg adduces evidence from ancient Egyptian, Sumerian and Persian documents. We shall now present the Persian evidence; the Sumerian and Egyptian number notations will be discussed later.



View of columns I–IV of the Persian text at Behistûn from the ledge below the inscriptions (from [illegible]*g, L. W. (1907)* The Sculptures and inscriptions of Darius the Great on the Rock of Behistûn in Persia, *British Museum).*

In the Behistûn inscription of Darius the Great (*c.* 500 BC) we find the number signs reproduced below.

The old Persian number notation is quite similar to the Babylonian method of writing numbers up to 10 (shown below).

Exercise 5

What are the similarities and differences between the Persian and Babylonian number signs shown above? Suggest a possible reason for the differences.

SA 5

The signs for 1 and 10 are just the same, and in both systems the numbers from 2 to 9 are written by repeating the simple 'wedge' for 1. However, in Babylonia the 'wedges' were placed in horizontal groups of 3, whereas in the Persian system they were arranged in vertical groups of 2. (For a reason for the differences see below.)

The scribes who wrote the Persian text of the inscription were familiar with Babylonian cuneiform script and adapted it to the Old Persian language. Now the question arises: Why did they change the form of the number signs for 7, 8 and 9, arranging the wedges in pairs? The Babylonian sign for 9 can be recognized at a glance whereas in reading the Persian signs for 8 and 9 we have to count the wedges in the first or second row. Why did the Persian scribes replace the perfect Babylonian sign for 9 by a less practical arrangement of the wedges in pairs? The most plausible hypothesis would seem to be that they adapted the Babylonian number notation to their own counting habit, *counting by pairs.* This is the hypothesis which Seidenberg puts forward. It is, of course, true that the Persian number words of the time were purely decimal and show no trace of a 2-system. Nevertheless, their earliest counting habit may well have been counting by pairs.

Part of Babylonian clay tablet (British Museum).

N1.4 NEO-2-COUNTING

In pure 2-count, 6 and 8 are pronounced as

$$6 = 2 + 2 + 2 \text{ and } 8 = 2 + 2 + 2 + 2$$

This is very inconvenient. Another method, which is similar but more efficient, uses special words for 3 and 4 and pronounces 6 and 8 as 'twice three' and 'twice four'. Thus, the Toba in Paraguay have the following number words[1]:

1	*nathedac*
2	*cacayni* or *nivoca*
3	*cacaynilia*

Toba tribesmen outside their home (Musée de l'Homme, Paris).

1 From Kluge, Theodor (1939) *Zahlenbegriffe der Völker Amerikas, Nordeurasiens, der Munda, und der Palaioafrikaner*, Berlin, p. 223.

4 *nalotapegat*
5 = 2 + 3 *nivoca cacaynilia*
6 = 2 × 3 *cacayni cacaynilia*
7 = 1 + (2 × 3) *nathedac cacayni cacaynilia*
8 = 2 × 4 *nivoca nalotapegat*
9 = (2 × 4) + 1 *nivoca nalotapegat nathedac*
10 = 2 + (2 × 4) *cacayni nivoca nalotapegat*

Seidenberg calls this method *neo-2-counting*. It has several variants, which may well be independent inventions. All these variants have in common the decompositions

$$6 = 2 \times 3 \text{ (or } 3 + 3)$$

$$8 = 2 \times 4 \text{ (or } 4 + 4)$$

but 7 and 9 are sometimes expressed as

$$7 = 6 + 1 \text{ and } 9 = 8 + 1$$

and in other cases as

$$7 = 8 - 1 \text{ and } 9 = 10 - 1$$

or even as

$$7 = 4 + 3 \text{ and } 9 = 5 + 4$$

On Map 1 (p. 16) neo-2-counting is indicated by circles, no distinction between variants being made.

Exercise 6

Comment briefly on the distribution of neo-2-count. What hypothesis does this distribution suggest?

SA 6

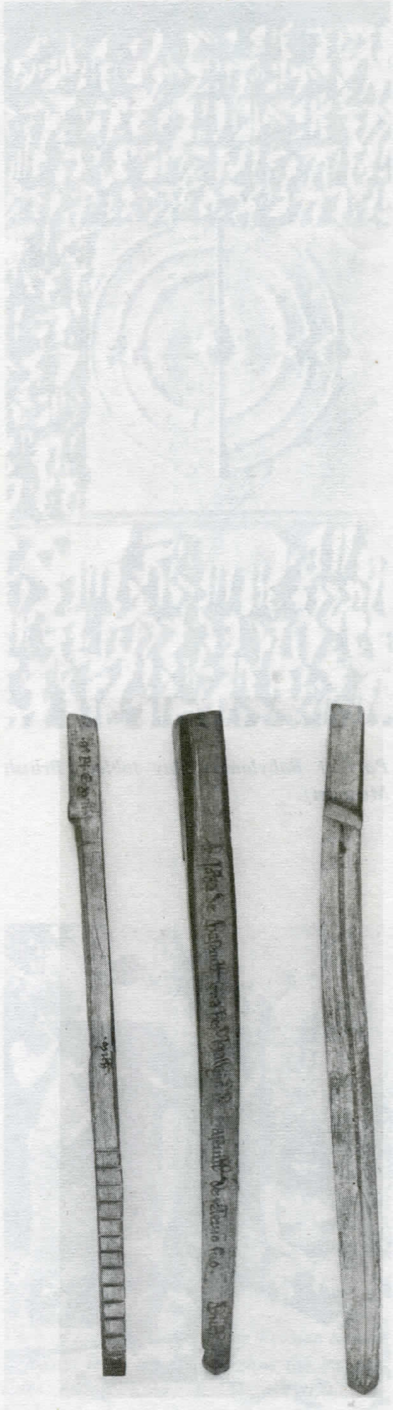
Neo-2-count is widespread in Africa, southern Asia, Australia and North and South America. In Australia, New Guinea and South America the neo-2-count regions are adjacent to pure 2-count regions. A possible hypothesis is that all were originally regions of pure 2-count but that the pure 2-count was subsequently replaced in most areas by the more efficient neo-2-count.

So we see from the map that neo-2-count is widespread in Africa, southern Asia, Australia and the Americas. In South America the expressions $7 = 6 + 1$ and $9 = 8 + 1$ prevail (see the example of the Toba quoted above) whereas in Africa divisions into nearly equal parts, such as

$$7 = 4 + 3 \text{ and } 9 = 5 + 4$$

are quite frequent. We also see that in Australia and New Guinea as well as in South America the regions of neo-2-count are adjacent to areas of pure 2-count. To explain this, Seidenberg assumes that all these regions originally had pure 2-count, and that later on this very inconvenient counting method was replaced by the more efficient methods of neo-2-count.

Can we explain the development of neo-2-count from pure 2-count? It might seem that the decompositions $6 = 3 + 3$ and $8 = 4 + 4$ are totally different from $6 = 2 + 2 + 2$ and $8 = 2 + 2 + 2 + 2$. Still one can imagine a passage from pure-2-count to neo-2-count through the intermediate of tally sticks or written numbers. Let us suppose that some pure 2-counting people would memorize and transmit numbers by carving notches in a tally stick. Since they counted by pairs, their carved numerals



Thirteenth-century Exchequer tallies (Crown copyright, Science Museum, London).

would probably look like this:

$$6\ \begin{matrix}|\,|\,|\\|\,|\,|\end{matrix}\qquad 7\ \begin{matrix}|\,|\,|\,|\\|\,|\,|\end{matrix}\qquad 8\ \begin{matrix}|\,|\,|\,|\\|\,|\,|\,|\end{matrix}$$

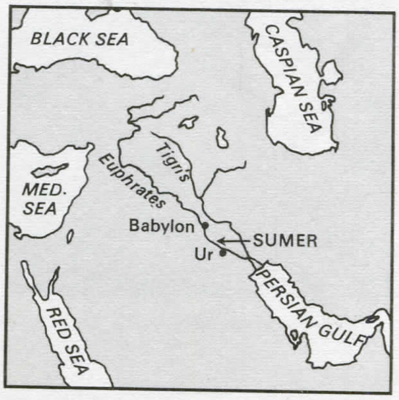
This is not pure fancy for numerals of just this kind were actually written by the Egyptians as early as 3000 BC as will be shown in Unit N3. The earliest Sumerian numerals, written on clay, were of the same type as we shall see presently.

Now suppose that carved or written numerals of this kind were used in ancient times by a 2-counting population, and that this method of noting numbers was taken over by neighbours who had independent number words for 1, 2, 3 and 4. They might read the sign for 6 as 3 + 3, the sign for 7 as 4 + 3 and the sign for 8 as 4 + 4. In this way the passage from pure 2-count to neo-2-count might be explained.

Exercise 7

Make your own brief summary of the explanation of the passage from pure to neo-2-count given above.

Seidenberg also discusses the Sumerian and Egyptian methods of writing numbers, which were in use about 3000 BC, and he regards them as giving evidence for the existence of 2-count or neo-2-count in the Near East about 3000 BC. The evidence is not conclusive but we shall consider the facts. The figure below is a reproduction of a Sumerian clay tablet dating from about 3000 BC or earlier.



The location of Sumer.

Sumerian clay tablet (Staatliche Museen zu Berlin, DDR).

In the middle column of the tablet we find (from bottom to top) the numbers

1, 2, 3, 4, 5, –, 7, 8, 9 (6 is broken off)

From 4 upwards the numbers are divided into equal or nearly-equal parts, as in a neo-2-system:

$$4 = 2 + 2,\ 5 = 3 + 2,\ 7 = 4 + 3,\ 8 = 4 + 4,\ 9 = 5 + 4$$

Essentially the same method of writing numbers is found in Ancient Egyptian texts. Here too the numbers below 10 are formed by repeating the sign for 1—a vertical stroke—as often as necessary as shown below.

<table>
<tr><td>|</td><td>||</td><td>|||</td><td>||
	or				</td><td>		
	</td><td>		
		</td><td>			
		</td><td>			
			</td><td>				
			or		
		</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
</table>

Exercise 8

List the neo-2- divisions indicated by the Egyptian number signs above.

SA 8

$4 = 2 + 2$
$5 = 3 + 2$
$6 = 3 + 3$
$7 = 4 + 3$
$8 = 4 + 4$
$9 = 5 + 4$

Seidenberg's hypothesis is that this variant of the neo-2-system spread out from the ancient civilizations of Mesopotamia and Egypt towards the middle zone of Africa, where it is still found today.

N1.5 4-COUNT AND 12-COUNT

The two counting methods just explained, 2-count and neo-2-count, do not carry us very far; beyond 10 they become impracticable.

A slightly better system is *4-count*—counting in groups of four. This system is not very common. Seidenberg gives two examples from Africa. The number words of the Afudu and Huku are equivalent to:

Afudu	Huku
$6 = 4 + 2$	$8 = 2 \times 4$
$7 = 4 + 3$	$9 = 2 \times 4 + 1$
$8 = 4 \times 2$ or $4 + 4$	$13 = 12 + 1$
$9 = 4 + 5$	$14 = 12 + 2$
	$15 = 12 + 3$
	$16 = (2 \times 4) \times 2$
	$17 = (2 \times 4) \times 2 + 1$
	$18 = (2 \times 4) \times 2 + 2$
	$19 = (2 \times 4) \times 2 + 3$

In South America, the Lulu count by the 4-system at least to 9 and the Charrua and Mocovi at least to 8.

Exercise 9

What is special about the number 12 in the counting method of the Huku?

SA 9

It appears to be used as a 'stepping-stone' from which counting restarts.

So we see from the number words of the Huku that in their counting the number 12 is used as a stepping-stone from which the counting starts anew. In other languages also the number 12 plays a special role. We often count by the *dozen*; the word is derived from the French word *douzaine*, which was actually an alternative term for the French shilling, the *sou*. In France, the old table of monetary equivalents was:

1 *livre* = 20 *sou*
1 *sou* = 12 *denier*

(This was derived from Charlemagne's monetary standard of 780, which embodied a basic 12-unit:

1 *libra* or *talentum* = 20 *solidus*
1 *solidus* = 12 *denarius*)

Obverse and reverse of coins of Charlemagne, solidus *and* denarius *(British Museum).*

The *sou*, being equivalent to 12 *denier*, thus came also to be called a *douzaine* (a 'twelver'). We call a dozen dozens a *gross*; this word is derived from the French *grosse douzaine* (a 'strong dozen'). The dozen and the gross are both still used for counting, chiefly in connection with commercial transactions. We also know, for example, that the Romans divided their main weight, the *as*, into 12 *unciae* (or ounces) and their measure of length, the *pes* (foot), into 12 *pollices* (inches).

Exercise 10

Suggest a possible reason for the special role of the number 12 in connection with weights and measures.

SA 10

The most obvious reason is its exact divisibility into half, quarter and third.

In addition to its divisibility, there is another possible reason for the special role of 12 as a measure of quantity. This relates to the idea of 'excess' of which we have examples in the *1001 Arabian Nights* and the French use of *quinze jours* for 'fortnight'. Although these two examples illustrate an excess of 1 only, it is possible to produce evidence suggesting the existence of cases of an excess of 2. Thus, the special role of 12 cannot be used to defeat the argument that *counting by tens is 'natural' because of the ten fingers on a man's hands.*

There is also much evidence to suggest that the number 12 has often had mystical or special symbolic significance. We can think of the twelve Christian apostles, the twelve ships of each of the heroes Ajax and Odysseus in Homer's *Iliad* and so on.[1]

N1.6 THE BINARY SYSTEM

In our decimal system numbers are expressed as sums of powers of ten with coefficients ranging from zero to nine. Thus, if we write 209, we mean

$$2 \times 10^2 + 0 \times 10^1 + 9 \times 10^0 \quad (\text{Note that } 10^0 = 1)$$

Instead of ten, any other positive integer (e.g. 2 or 5) might have been used as a base of the system. Of special interest, because it is used in electronic digital computers, is the *binary* or *dyadic system* which has 2 as its base.

1 For a fuller discussion, see Menninger, *op. cit.*

In this system, numbers are expressed as sums of powers of 2 (including $2^0 = 1$) with coefficients 0 or 1, for instance

$$13 = 8 + 4 + 1 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

The coefficients of the successive powers of 2 are 1, 1, 0, 1. Hence the number 'thirteen' is written in the binary system as 1101.

The procedure to express any given number N as a sum of powers of 2 is as follows. Let 2^a be the highest power of 2 less than or equal to N. Subtract 2^a from N. The difference $N - 2^a$ is less than 2^a. If the difference is zero, N is just 2^a. If not, repeat the procedure, again subtracting the largest possible power 2^b, and so on until the difference becomes zero. At the end we obtain

$$N = 2^a + 2^b + \cdots + 2^i$$

For instance, if N is 22, we obtain

$$\begin{aligned} 22 &= 16 + 4 + 2 \\ &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \end{aligned}$$

hence the binary notation for 'twenty-two' is 10110.

There is no evidence to suggest that the binary system was used to form number words. It is easy to find a reason for this: the base 2 is too small for everyday practical use. The decomposition of 30, for example, into 16 + 8 + 4 + 2 is too cumbersome. However computers use the binary system for arithmetic calculations.

Computer terminal (BBC).

The 'memory' of a computer can retain only successions of zeros and ones, nothing else. If we press the button marked 5 on a computer, it immediately transforms the number 5 into the sequence 1–0–1 according to a transformation rule fixed once and for all.

Press button marked 1, the computer notes 1
press button marked 2, the computer notes 1–0
press button marked 3, the computer notes 1–1

and so on.

Now what happens if we feed a higher number, say 240, into the computer?

First we press the button marked 2; the computer notes the binary equivalent 1–0. Next we press the button marked 4. The computer now automatically multiplies the number 1–0 (= two) just noted by 1–0–1–0 (= ten) which gives 1–0–1–0–0 (i.e. twenty) and adds 1–0–0 (= four). The result of this addition in the binary system is 1–1–0–0–0 ($= 2^4 + 2^3 = 24$) which is correct because at this stage we have fed 24 into the computer. Finally we press the button marked 0. The computer again multiplies the result of the preceding operations, i.e. 1–1–0–0–0, by 1–0–1–0 (= ten because ten is the base of our system) and it adds 0. Thus, the computer has in binary notation

1–1–0–0–0 × 1–0–1–0 = 1–1–1–1–0–0–0–0

which means in decimal notation

$$24 \times 10 = 240$$

You should check the correctness of the result:

$$2^7 + 2^6 + 2^5 + 2^4 + 0 + 0 + 0 + 0 = 128 + 64 + 32 + 16 = 240$$

Exercise 11

Distinguish between *pure 2-count* and the *binary system* of numbers.

Pure 2-count is a system of counting having number words for 1 and 2 only. Words for numbers greater than two are formed compositely. Thus 3 is called 'two-one', 4 is called 'two-two', and so on. The *binary system* of numbers, on the other hand, is a place-value system of recording numbers used in computers and similar to our own decimal place-value system, but having the base 2 instead of the base 10. There is no evidence that a binary counting system has ever existed.

N1.7 THE 5–10- AND 5–20-SYSTEMS

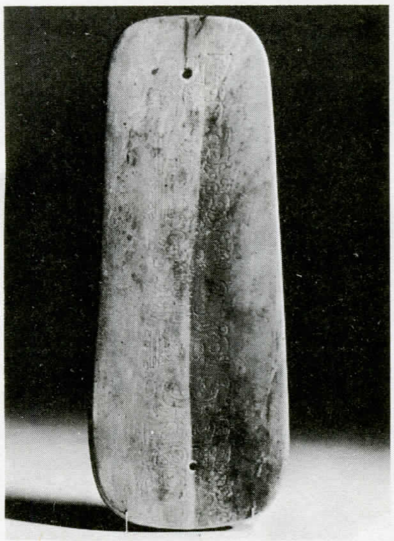
In many parts of the earth, counting by independent number words stops at 5. Between 5 and 10, composite number words are used. Thus, for example, the Aztecs of Mexico count as follows:

1 *ce*, hence 6 *chica-ce* (6 = 5 + 1)
2 *ome*, hence 7 *chic-ome* (7 = 5 + 2)
3 *yey*, hence 8 *chicu-ey* (8 = 5 + 3)
4 *naui*, hence 9 *chic-naui* (9 = 5 + 4)
5 *macuilli*

Aztec calendar (*Musée de l'Homme*).

In the composite number words for 6, 7, 8 and 9 the first part *chica* obviously means either 'five' or 'one hand' or something like this, although *chica* cannot be derived from *macuilli* = 5.

This method of counting, which is widespread among the indigenous peoples of the Americas, Asia and Africa, may be called *5-count*. It can easily be extended to numbers between 10 and 20. Thus the Aztecs continue their count as follows:



The 'Plaquette de Leyde', showing the first deciphered Mayan date, AD *320 (Musée de l'Homme).*

10 *matlactli*
11 *matlactli-on-ce* $(11 = 10 + 1)$
12 *matlactli-on-ome* $(12 = 10 + 2)$
13 *matlactli-on-yey* $(13 = 10 + 3)$
14 *matlactli-on-naui* $(14 = 10 + 4)$
15 *caxtulli*
16 *caxtulli-on-ce* $(16 = 15 + 1)$
etc. until
19 *caxtulli-on-naui* $(19 = 15 + 4)$

Beyond 20 there are two methods of extending the system, both of which occur frequently. Following Seidenberg we may call these methods the *5–10-system* and the *5–20-system.*

In the 5–10-system multiples of ten are constructed simply as multiples of ten not as multiples of twenty. For instance in the 5–10-system 80 is just 8 times ten and not *quatre-vingt* and 90 is 9 times ten and not *quatre-vingt-dix*. The 5–10-system is mainly found in Africa and North America (see Map 3 p. 42) and is very often connected with finger-counting. In many cases the number words are derived from the names of the fingers.

In the 5–20-system numbers like 40 and 60 are pronounced as 2×20 and 3×20. The Aztecs provide a typical example of a fully developed 5–20-system. Their number-words for 20, 30, etc. are as follows:

20 *cem-poualli* $(20 = 1 \times 20)$
30 *cem-poualli-om-matlactli* $(30 = 1 \times 20 + 10)$
40 *ome-poualli* $(40 = 2 \times 20)$
100 *macuil-poualli* $(100 = 5 \times 20)$

There are also special words for powers of 20:

$20^2 = 400$ *cen-tzuntli*
$20^3 = 8000$ *cen-xiquipilli*

A closely related system is that of the Mayas of Yucatan. Their number words up to 19 are as follows:

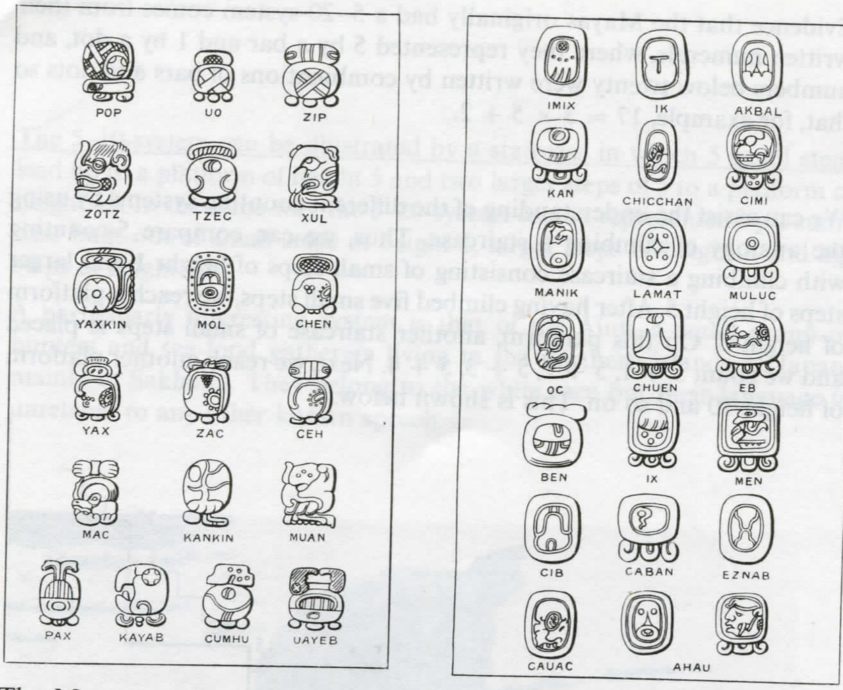
1 *hun*	9 *bolon*
2 *ca*	10 *lahun*
3 *ox*	11 *buluc*
4 *can*	12 *lah-ca* $(10 + 2)$
5 *ho*	13 *ox-lahun* $(3 + 10)$
6 *uac*	14 *can-lahun* $(4 + 10)$
7 *uuc*	15 *ho-lahun* $(5 + 10)$
8 *uaxac*	. . .
	19 *bolon-lahun* $(9 + 10)$

The Mayas have special words for powers of 20:

(1 *hun*)
20 *hun-kal*
20^2 *hun-bak*
20^3 *hun-pic*
20^4 *calab*

The words for multiples of 10 below 400 are as follows:

30 *lahu-cakal* $(2 \times 20 - 10)$	100 *ho-kal* (5×20)
40 *ca-ikal* (2×20)	120 *uac-kal* (6×20)
50 *lahu-y-oxkal* $(3 \times 20 - 10)$	140 *uuc-kal* (7×20)
60 *oxkal* (3×20)	200 *lahun-kal* (10×20)
70 *lahu-cankal* $(4 \times 20 - 10)$	300 *ho-lhu-kal* (15×20)
80 *can-kal* (4×20)	
90 *lahu-y-hokal* $(5 \times 20 - 10)$	



Mayan day and month signs (British Museum).

The Mayan number sequence is unusual and almost certainly did not arise from the everyday needs of the common people. Menninger[1] suggests that it was an artificial creation of the priests, possibly intended for calendar computations.

Unlike the Aztecs the Mayas have independent number words for 6, 7, 8 and 9, that is they do not decompose 6 into 5 + 1, 7 into 5 + 2, etc. But their method of *writing* numbers clearly shows that they had a 5–20-system. They represented 5 by a bar and 1 by a dot. Numbers below 20 were represented by bars and dots, e.g. 17 by three bars and two dots:

17

For 20 the Mayas had a special sign:

This sign with, for example, two or three dots meant $2 \times 20 = 40$, $3 \times 20 = 60$, and with one or two bars it meant $5 \times 20 = 100$, $10 \times 20 = 200$ and so on[2].

From this we may conclude that the Mayas once had a complete 5–20-system just as did their neighbours in Mexico.

Exercise 12

Distinguish between *5–10-* and *5–20-count*, and summarize the principal evidence from which we deduce the nature of the counting systems of the Aztecs and Mayas.

SA 12

5–10-count has independent number words up to 5 followed by composite number words from 6 to 9, and words for multiples of 10 are constructed explicitly as such. *5–20-count* extends the 5-count principle to numbers up to 19, and even multiples of 10 are expressed as 2×20, 3×20, and so on. The Aztecs count 1, 2, 3, 4, 5, 5 + 1, 5 + 2, 5 + 3, 5 + 4, 10, 10 + 1, . . . , 15, 15 + 1, . . . , 20, . . . , $1 \times 20 + 10$, . . . , 2×20, . . . , 5×20, etc., and there are special words for powers of 20. This is a typical 5–20-system. The Mayas, on the other hand, have independent number words for 6, 7, 8 and 9 but form their number words for even multiples of 10 in a manner similar to the Aztecs and they also have special words for powers of 20.

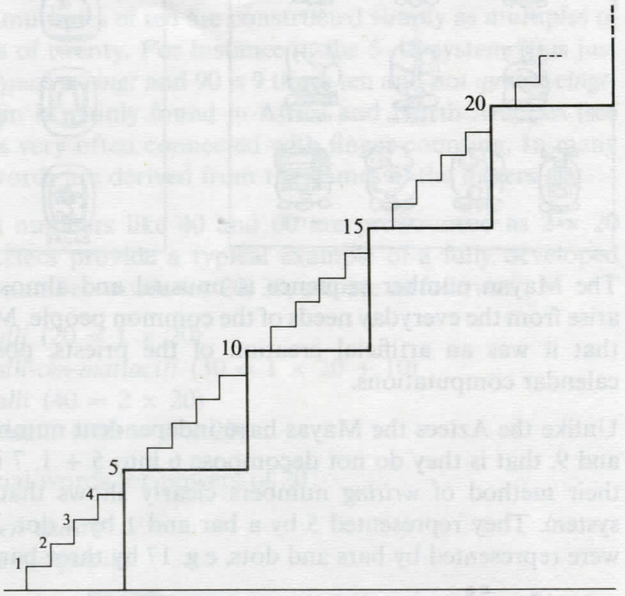
Mayan numbers on stone fragment found in British Honduras (British Museum).

1 *Op. cit.*, p. 61.

2 We shall discuss the Mayan numerals in greater detail in Unit N3.

Evidence that the Mayas originally had a 5–20-system comes from their written numerals, where they represented 5 by a bar and 1 by a dot, and numbers below twenty were written by combinations of bars and dots so that, for example, $17 = 3 \times 5 + 2$.

We can assist the understanding of the different counting systems by using the analogy of climbing a staircase. Thus, we can compare 5-counting with climbing a staircase consisting of small steps of height 1 and larger steps of height 5. After having climbed five small steps, we reach a platform of height 5. On this platform, another staircase of small steps is placed and we count $5 + 1, 5 + 2, 5 + 3, 5 + 4$. Next, we reach another platform of height 10 and so on. This is shown below.

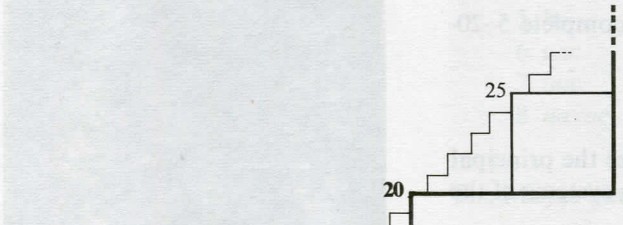


Staircase of 5-count.

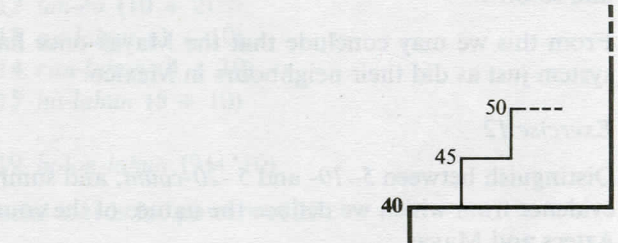
Exercise 13

Draw staircases representing the 5–10- and 5–20-systems

SA 13



Staircase of 5–10-system.



Staircase of 5–20-system.

Note Unit steps have been omitted after 5.

The 5–10-system can be illustrated by a staircase in which 5 small steps lead up to a platform of height 5 and two larger steps of 5 to a platform of height 10. In the same way the 5–20-system can be represented by a staircase built out of small steps of height 1, larger steps of height 5 and big steps of height 20.

A particularly interesting system is that of the Ainu, a small group of hunters and sea-food gatherers living in the northern islands of Japan, mainly in Sakhalin. They belong to the white race but their language is unrelated to any other known speech.

Ainu chieftain (Japanese National Tourist Organization, London).

The Ainu have just seven number words from which all others are combined, namely

1 *shi-ne*
2 *tu*
3 *re*
4 *ine*
5 *ashik-ne*, meaning 'hand'
10 *wan*, meaning 'two-sided' (two hands)
20 *hot-ne*, derived from *hot* = 'the whole' and *ne* = 'to be'

The numbers from 6 to 9 are obtained by subtraction:

$6 = 10 - 4$ *i-wan* (4 from 10)
$7 = 10 - 3$ *ar-wan* (3 from 10)
$8 = 10 - 2$ *tu-pesan* (2 steps down)
$9 = 10 - 1$ *shine-pesan* (1 step down)

The last two expressions show that the Ainu actually thought of counting as climbing a staircase.

Beyond 20, the counting goes on as follows:

$30 = 2 \times 20 - 10$	*wan e tu hotne*
$40 = 2 \times 20$	*tu hotne*
$50 = 3 \times 20 - 10$	*wan e re hotne*
$60 = 3 \times 20$	*re hotne*
$70 = 4 \times 20 - 10$	*wan e ine hotne*
$80 = 4 \times 20$	*ine hotne*
$90 = 5 \times 20 - 10$	*wan e ashikne hotne*
$100 = 5 \times 20$	*ashikne hotne*

$110 = 6 \times 20 - 10$ *wan e iwan hotne*
$120 = 6 \times 20$ *iwan hotne*

$200 = 1 \times 10 \times 20$ *shine wan hotne*
$300 = 5 \times 20 + 1 \times 10 \times 20$ *ashikne hot-ikashama shine wan hotne*
$400 = 2 \times (1 \times 10 \times 20)$ *tu-shine wan hotne*
$500 = 5 \times 20 + 2 \times (1 \times 10 \times 20)$ *ashikne hot-ikashama-tu-shine wan hotne*

and so on until

$1000 = 5 \times (1 \times 10 \times 20)$ *ashikne-shine wan hotne*

This is a very complicated way of forming number words, described by Menninger[1] to be 'a truly Cyclopean tower of numbers'.

Exercise 14

Which type of counting system is represented by the Ainu number words?

SA 14

5–20-count.

The 5–20-system has certain marked differences from our own decimal system. Five small steps of height 1 lead up, as we can see from the staircase on p. 28, to an intermediate platform of height 5 and four steps of height 5 lead to a platform of height 20. The next factors are again 5 and 4 and so on. It is found in many parts of the world (see Map 3, p.42). Later we shall see that it is closely connected with *finger-and-toe counting*.

Exercise 15

(a) By reference to the appropriate staircases, compare our own decimal counting system with the 5–20-system, and suggest why the latter can be said to be 'alternating'.
(b) What connection between 5–20-count and finger-and-toe counting suggests itself?

SA 15

(a) Our decimal system is a perfectly regular counting system with base 10. Ten small steps of height 1 lead up to a platform of height 10, ten large steps of height 10 to a still larger platform of height 100, and so on. The next step is always 10 times the preceding one. The 5–20-system is different. Its fundamental factors are alternately 5 and 4. The heights of the successive platforms are:

$$5,\ 20,\ 5 \times 20,\ 20^2,\ 5 \times 20^2,\ 20^3, \ldots$$

It can be called an *alternating system* because of its alternating factors.
(b) The first step 5 was chosen because we have five fingers on each hand and 5 toes on each foot. We have two hands and two feet, making 4 groups of 5, all in all a total of 20 fingers and toes.

---

1 *Op. cit.*, p. 69.

N1.8 THE 10–60-SYSTEM

Another alternating system is the *10–60-system.* You will not find it on Map 3 because it is no longer used. It was used by the Sumerians, who reached a high level of civilization in Southern Babylonia (near the Persian Gulf, where the rivers Euphrates and Tigris come together) in the third millennium BC (see map, p. 21). Their number words below 60 were formed as follows:

1 *ash* or *gesh*
2 *min*
3 *esh*
4 *limmu*
5 *ia*
6 *ash* (from *ia-ash* $= 5 + 1$)
7 *imin* (from *ia-min* $= 5 + 2$)
8 *ussu*
9 *ilimmu* (from *ia-limmu* $= 5 + 4$)
10 *u*
20 *nish*
30 *ushu* (from *esh-u* $= 3 \times 10$)
40 *nin* (from *ni-min* $= 20 \times 2$)
50 *ninu* (from *ni-min-u* $= 20 \times 2 + 10$)

Up to this point we have a normal 10-system with traces of 5–20-count—nothing unusual. However, when we come to 60, a completely new phenomenon occurs: the number 60 is considered as a new unit and denoted by the same word *gesh*, which also can have the meaning 'one'. All higher steps are multiples of 60:

60	*gesh*
2×60	*gesh-min*
3×60	*gesh-esh*
10×60	*gesh-u*
60×60	*shar*
10×60^2	*shar-u*
60^3	*shar-gal*

The word *shar-gal* means 'big-shar'. From the formation of this word we may conclude, with a certain degree of probability, that *shar* $= 60^2$ was at one time a limit of counting and that at a later time the limit was extended to 60^3.

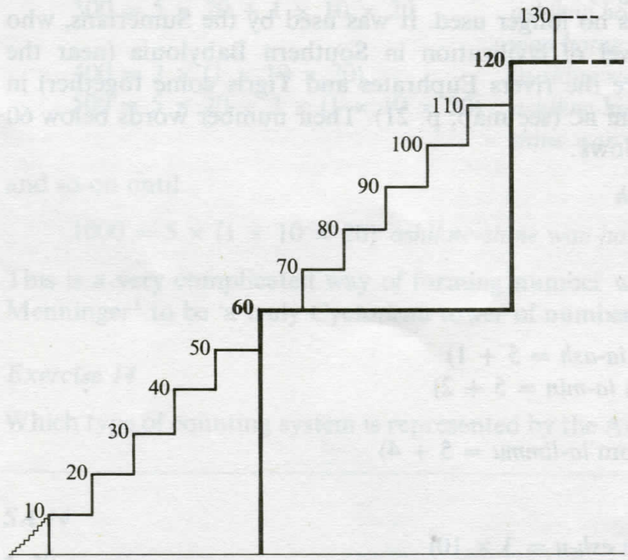
If we consider the system as a whole it becomes clear that it is a mixture of two quite different systems: an older 5–20-system, which was retained for numbers below 60, and a more recent (but still very old) 10–60-system, in which the stepping-stones were

$$1, 10, 60, 10 \times 60, 60^2, 10 \times 60^2, 60^3$$

(The staircase of the 5–20-system is shown on p. 28.)

Exercise 16

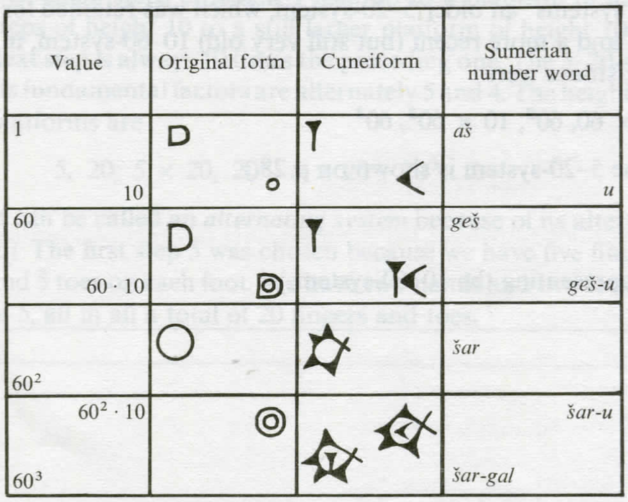
Draw a staircase representing the 10–60-system.



*Staircase of 10–60-system (*Note *Unit steps have been omitted after 10).*

In the written numerals of the Sumerians we find no trace of a 5–20-system. We are bound to conclude that the 5–20-system was developed as a system of spoken number words before the Sumerians learned to write numbers. On the other hand it is quite possible that the 10–60-system was first developed as a system of writing numbers and that the number words for 60, 10 × 60, etc. were formed under the influence of the writing system.

In any case the written numerals of the Sumerians correspond exactly to the number words just mentioned. The signs were impressed in a clay tablet by means of a stylus. Originally, a round stylus was used for writing numbers. If the stylus was impressed vertically a circular impression resulted, which meant 10, and if it was put in a slanted position a semicircular sign was obtained which meant 1 (see Sumerian tablet, p. 21). A larger semicircle, drawn by a sharp stylus, meant 60 and a larger circle meant 60^2. By combining the symbols for 10 and 60 the number 60 × 10 was written and, by combining the signs 60 and 60^2, 60^3 could be written (see column 3 of the diagram below[1]).



Early Sumerian numerals and number words (third millennium BC*).*

1 Adapted from Menninger, *op. cit.*, p. 164.

Later on the number 1 was written as a wedge and 10 as an L-shaped wedge. The number 60 was also written as a single wedge, thus expressing the idea that 60 was regarded as a new unit. The circle for 60^2 was replaced by a square formed by four wedges and so on (column 3).

This choice of 60 as a new unit and base of the system is unique in history. After 2000 BC the Babylonians and Assyrians took it over from the Sumerians together with cuneiform script.

Now two questions arise. First, what led to the choice of 60 as the next stage after 1 and 10? Second, why was 60 considered as a 'big unit' and represented by the same symbol as 1?

O. Neugebauer discusses these problems as follows[1]:

> The question of the origin of the sexagesimal system is inextricably related to the much more complex problem of the history of many concurrent numerical notations and their innumerable local and chronological variations.
>
> But it is not enough to realize that the 60-division is only one of several contemporary norms between higher and lower units. The essential point lies in the use of the place value notation, regardless of the value of the ratio between consecutive units. No historical theory of the origin of the sexagesimal system is acceptable if it does not account also for this extraordinary feature, namely, the use of the same small number of symbols for different values, depending on the arrangement. A variety of 'bases' is well known from number words and number writing all over the world. The place value notation, however, is the most striking feature of the Babylonian system.
>
> A problem of this kind cannot be solved by speculation, but only by a systematic analysis of the written documents.

Neugebauer points out that the overwhelming majority of cuneiform texts are concerned with economic matters, and not with religion, magic or number mysticism. He continues:

> In economic texts units of weight, measuring silver, were of primary importance. These units seem to have been arranged from early times in a ratio of 60 to 1 for the main units 'mana' (the Greek μνα 'mina') and shekel. Though the details of this process cannot be described accurately, it is not surprising to see this same ratio applied to other units and then to numbers in general. In other words, any sixtieth could have been called a shekel because of the familiar meaning of the concept in all financial transactions. Thus the 'sexagesimal' order eventually became the main numerical system and with it the place value writing derived from the use of bigger and smaller signs. The decimal substratum however, always remained visible for all numbers up to 60. Similarly, other systems of units were never completely extinguished. Only the purely mathematical texts, which we find well represented about 1500 years after the beginning of writing, have fully utilized the great advantage of a consistent sexagesimal place value notation. Again, 1000 years later, this method became the essential tool in the development of a mathematical astronomy, whence it spread to the Greeks and then to the Hindus, who contributed the final step, namely, the use of the place value notation also for the smaller decimal units. It is this system that we use today.

Nothing is more natural, when one monetary unit is 60 times another, than denoting the larger unit again by 1 and its multiples by the ordinary symbols for 1, 2, . . . , 10, ... We all do it every day. 'Two-fifty' just means 2 pounds and 50 pence, or 2 dollars and 50 cents, as the case may be.

We shall be discussing the Sumerian system of numerals in greater detail in Unit N3, and we shall see that it quite naturally led to a place-value system for integers and fractions with the base 60.

Exercise 17

How does Neugebauer answer the two questions about the choice of 60 as the next stage after 10, and its representation by the symbol for 1 (see above)?

---

1 Neugebauer, O. (1957) *The Exact Sciences in Antiquity*, Brown University Press.

SA 17

Neugebauer answers the questions by considering 'economic units', which were arranged 'from early times in a ratio of 60 to 1'. 60 is then naturally regarded as a single large unit and represented by the same symbol as 1.

N1.9 REMNANTS OF 20-COUNT IN WESTERN EUROPE

Our modern Western languages are Indo-European, and our counting is purely decimal. Yet there are relics of a 20-system in it. The French, for example, pronounce 80 and 90 as *quatre-vingt* and *quatre-vingt-dix*, that is as 4×20 and $4 \times 20 + 10$. They also pronounce 70 as $60 + 10$. In English we have the expression 'three-score' for 60. In the familiar Authorized Version of the Bible we read:

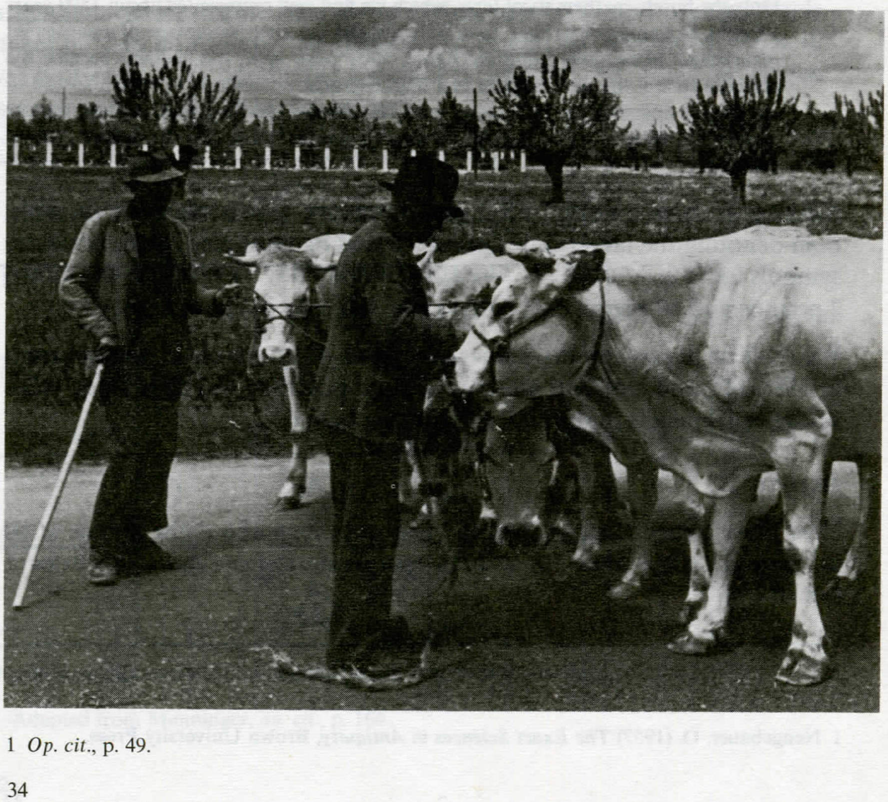
> The days of our years are three-score years and ten; and if by reason of strength they be fourscore years, yet is their strength labour and sorrow; for it is soon cut off, and we fly away. (Psalm 90:10)

Shakespeare also counts by scores.

> Why, this boy will carry a letter twenty miles as easy as a cannon will shoot point-blank twelve score. (*Merry Wives of Windsor*, Act II, Scene 2)

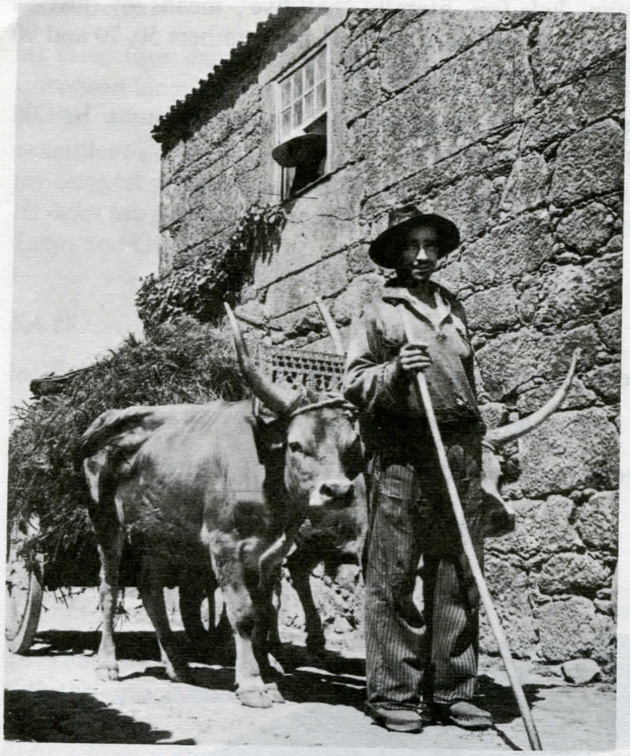
Menninger draws particular attention to examples from the everyday language of the common people. He writes[1]:

> It is difficult for an illiterate peasant to tell how many years he has lived; the Piedmontese, for instance, would answer that he carries *quat borla*, '4 large loads', on his bent back, meaning $4 \times 20 = 80$ years. This is far easier for him to visualize; he does not climb the steps of his years age, 77, 78, 79, 80, but remains standing on the ground and, like a mule, takes up the 'loads' of his years grouped in bundles of 20. The Sicilian dialect has no special word for the 20-group, so that in Sicily the peasant says *tri vintini et deci*, '3 twenties and 10', for 70 years; thus he uses the number word *vinti*. In some regions of Spain the monetary units (1 *duro* = 20 *reales*) are used in counting, so that a peasant there would reply that he is '3 duros and 10 reales' old.



Piedmontese in the fields at Cuneo, N. Italy (by courtesy of the Italian State Tourist Office).

1 *Op. cit.*, p. 49.



Portugese peasant from the province Beira Litoral (Camera Press, photograph Paul Almasy).

Another example of counting money in twenties can be found in Portugal, where the word *vintens*, 'twenties', is in use. Sometimes it is not actually pronounced: *dois cinco* is an abbreviation for *dois vintens e cinco* (2 × 20 + 5 = 45) and *seis menos cinco* means 'six twenties less five' (= 115).

Probably the oldest documentary evidence is an inventory of an English monastery, written about AD 1050, which contains expressions from vernacular language together with their Latin translations:

V	*scora scoep*	*quinquies viginti oves*	(5 × 20 sheep)
VIII	*scora oecer*	*octies viginti agri*	(8 × 20 acres)

In Sweden, farmers often say *fyra sneser* (four *snes*) when they mean eighty. In Holland some peasants count eggs by the *snees*, a *snees eieren* meaning 20 eggs. The word *snes* or *snees* (German *Schneise*) has the original meaning of a rod or twig used to line things up in a row. A Baltic monastic register from 1186 includes in its annual income 24 *snesas anguillarum* = 24 × 20 eels.

Another example of counting by twenties is found in the Danish language. Up to 40 the counting is decimal:

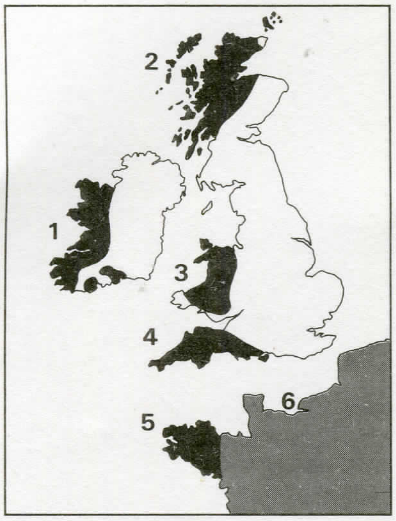
10 *ti*
20 *tyve*
30 *tredive*
40 *fyrretyve*

but 60 and 80 are considered as multiples of 20:

60 = 3 × 20 *tre-sinds-tyve*
80 = 4 × 20 *fir-sinds-tyve*

The expressions for 50, 70 and 90 are very curious:

$50 = 2\frac{1}{2} \times 20$ *halv-tre-sindstyve*
$70 = 3\frac{1}{2} \times 20$ *halv-fjerd-sindstyve*
$90 = 4\frac{1}{2} \times 20$ *halv-fem-sindstyve*



The areas in which Celtic was and is spoken; 1 Irish: 2 Gaelic in Scotland: 3 Welsh: 4 Cornish (extinct): 5 Breton: 6 Gallic (extinct).

In the last expression, *halv-fem,* literally 'half-five', means $4\frac{1}{2}$, just as *halb fünf* in German means 'half past four'. So the numbers 50, 70 and 90 were considered as half-integer multiples of 20.

In Celtic languages too traces of popular 20-count are found. In Old Irish we find a regular decimal system:

10	*deich*	60	*sesca*
20	*fiche*	70	*sechtmoga*
30	*tricha*	80	*ochtmoga*
40	*cethorcha*	90	*nocha*
50	*coica*	100	*cet*

But in popular speech expressions were used such as

deich mnaa a secht fichit

which means '10 women and seven twenties', i.e. 150 women.

Irish Aran islanders cleaning a currach (Irish Tourist Board, London).

This popular tendency to count by twenties has led to a regular 20-system in New Irish and in Scottish Gaelic. In New Irish 40 is pronounced as *da fichit* (2 × 20), 60 as *tri fichit* (3 × 20) and so on.

A similar development can be found in Welsh, Cornish and Breton, which are all closely related languages. The Welsh number words are as follows:

1	*un*	11	*un-deg-un*
2	*dau* (f. *dwy*)	12	*deuddeg* or *un-deg-dau*
3	*tri* (f. *tair*)	13	*un-deg-tri*
4	*pedwar* (f. *pedair*)	14	*un-deg-pedwar*
5	*pump*		......
6	*chwech*	20	*ugain*
7	*saith*	30	*tri-deg* or *deg-ar-hugain*
8	*wyth*	40	*pedwar-deg* or *deugain*
9	*naw*	60	*chwe-deg* or *trigain*
10	*deg*	80	*wyth-deg* or *pedwar-ugain*

Here we see in the alternative forms of 30, 40, 60, 80, number words equivalent to 10 + 20, 2 × 20, 3 × 20, 4 × 20 respectively.

It is of particular interest to note that, although French is a Romance language (derived from Latin), the expression *quatre-vingt* is not itself derived from a Latin equivalent. Also, the New Irish *da fichit* is not derived from an equivalent form in Old Irish.

Exercise 18

(a) Give four examples of the relics of 20-count in different Western European languages other than English and French.
(b) To what particular point is Menninger drawing attention with his examples?
(c) Suggest a hypothesis which explains why forms like *quatre-vingt* and *da fichit* are present in French and New Irish, although they are absent in Latin and Old Irish.

SA 18

(a) Examples are:

quat borla	('eighty years')—Piedmont
tri vintini	('sixty years')—Sicily
dois vintens	('forty')—Portugal
snees eieren	('twenty eggs')—Holland
deugain	('forty')—Wales
da fichit	('forty')—Ireland
etc.	

(b) That expressions involving twenties are especially to be associated with the vernacular.
(c) A possible hypothesis is that the practice of counting by twenties derives from a more ancient non-Indo-European language.

The hypothesis that the popular tendency to count by twenties is derived from a more ancient language, which was spoken in Western Europe before the Celtic period and which did not belong to the Indo-European language family, finds confirmation in the language of the Basques which is not related to the Indo-European family. The Basque words for 1 to 9 and 10 to 90 are:

1	*bat*	10	*hamar*
2	*biga*	20	*hogoi*
3	*hirur*	$30 = 20 + 10$	*hogoi-ta-hamar*
4	*laur*	$40 = 2 \times 20$	*berrogoi*
5	*bortz*	$50 = 40 + 10$	*berrogoi-ta-hamar*
6	*sei*	$60 = 3 \times 20$	*hirur-hogoi*
7	*zazpi*	$70 = 60 + 10$	*hirur-hogoi-ta-hamar*
8	*zortzi*	$80 = 4 \times 20$	*laur-hogoi*
9	*bederatzi*	$90 = 80 + 10$	*laur-hogoi-ta-hamar*

The area of the Basques.

This evidence, together with evidence from certain other non-Indo-European languages, enables us to conclude that in Western Europe before the expansion of Indo-European languages a 20-system of counting was in use. Afterwards the number words were replaced by Indo-European number-words like *quatre* and *vingt* but the tradition to count by multiples of 20 was retained in many places.

We do not know whether the Basque 20-system originally was a 5–20-system. It seems possible that the word *sei* for 6 was taken over from a Romance language and that 6 was originally pronounced as 5 + 1. Another trace of Romance influence is the Basque word *milla* for 1000.

N1.10 FINGER COUNTING

2000 shown on the fingers from an illustration in Bede's work on finger counting (Biblioteca Ñacional, Spain).

Finger counting is a very natural procedure: we might assume that every group could discover it for itself. Yet if we examine the details of finger counting we see that there are traditions in finger counting which were passed on from one group to another.

The number words of the Australian Chaap wurrong tribe in Victoria from 4 to 50 have the following meaning:

4 'two and two'
5 'one hand'
6 'one finger, hand'
7 'one hand and two'
8 'one hand and three'
9 'one hand and two and two'
10 'two hands'
11 to 19 ten plus the symbol for 1 to 9
20 'one twenty'
30 'one twenty and ten'
40 'two twenties'
50 'two twenties and ten'

Exercise 19

What counting systems can be detected in the Chaap wuurong number words?

SA 19

2-count, 5-count, 20-count.

The 5–20-system is very often associated with finger counting or with counting on fingers and toes.

Seidenberg[1] writes:

> Returning to Central Brazil, we note that of the dozen or so groups that von den Steinen met, only two, the Bakairi and the Bororo, use the 2-system. Rather, an independent word exists for 5 [i.e. not composed of 2 + 2 + 1]; then 6, 7, 8, 9 are constructed from 1, 2, 3, 4 with some modification to show that the speaker is in the second pentad. These tribes use their fingers and toes in counting, and in most, thought not all of them, the word for 5 is related etymologically to 'hand'. 'Ten' is with some of them a new word, with others it contains (etymologically) 'the hands'. Then for 11, 12, 13, 14 one modifies 1, 2, 3, 4 with an addition referring to a 'foot'.
>
> In all, with one exception, 'feet' occurs in the word 'twenty'.

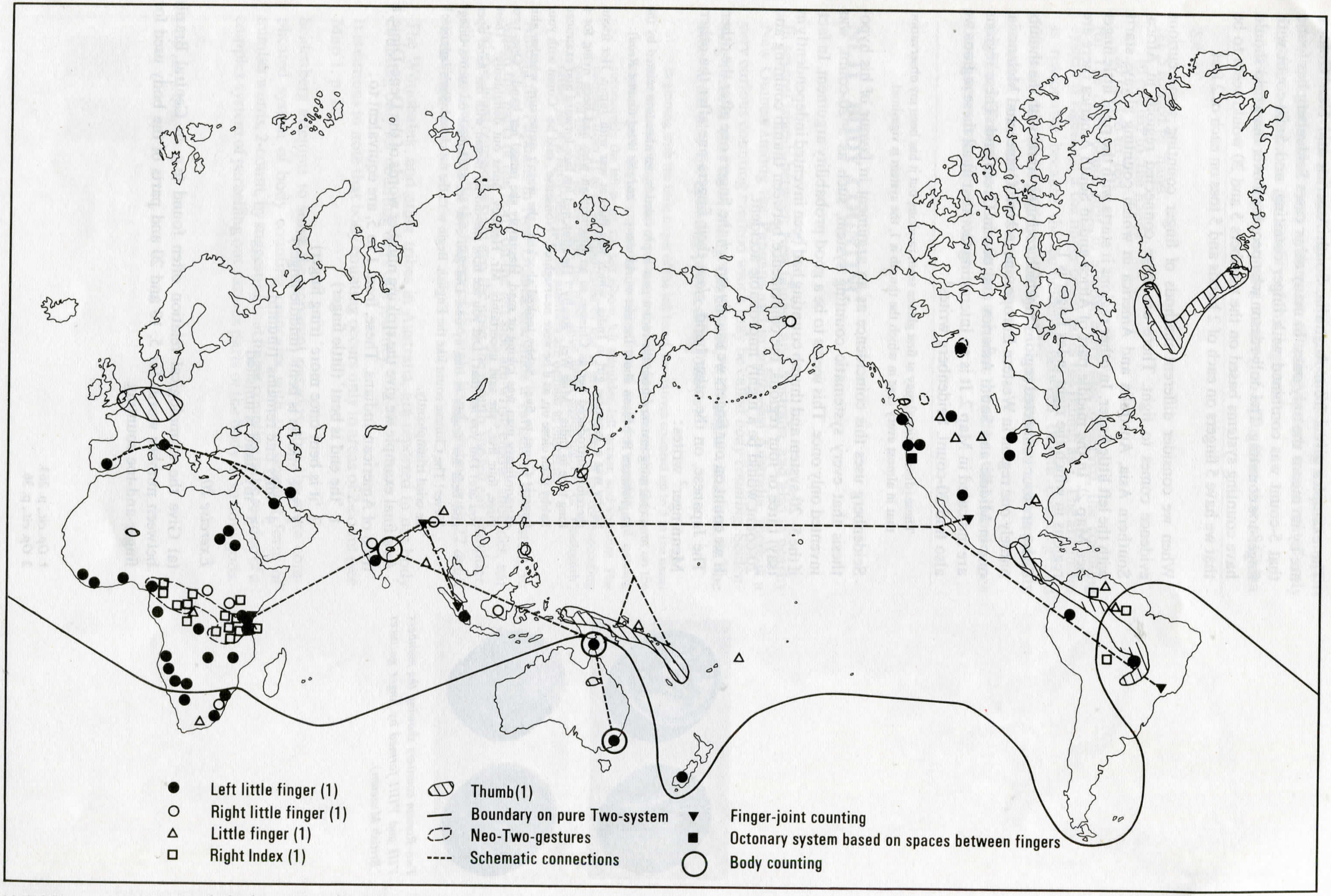
It is clear that we have here at least a basic fragment of a 5–20-system, and that this 5–20-system is closely connected with *finger-and-toe counting.*

The same phenomenon occurs in New Guinea. Menninger writes[2]:

> In translating the Bible for one tribe of Papuans, the passage (John5:5): 'And a certain man was there, which had an infirmity thirty and eight years' had to be expressed as 'a man lay ill one man (20), both sides (10), 5 and 3'. Even more picturesque is the expression for the number 99 in British New Guinea: 'four men die (80), two hands come to an end (10), one foot ends (5) and 4'.

In both cases we clearly have a 5–20-system, and the connection with finger-and-toe counting is equally clear.

1 *Op. cit.*, p. 225.
2 *Op. cit.*, p. 36.



MAP 2 Finger-counting practice (by kind permission of Professor A. Seidenberg).

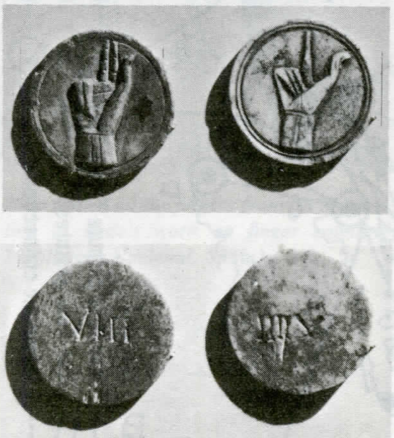
The examples given, from Australia, South America and New Guinea, are by no means the only ones. In many other cases Seidenberg has found that 5-count was combined with finger counting, and 5–20-count with finger-toe counting. The only reason why people all over the world should have counting systems based on the numbers 5 and 20 would seem to be that we have 5 fingers on each of 2 hands and 5 toes on each of 2 feet.

When we consider different methods of finger counting some curious evidence comes to light. There are large connected regions in Africa, Southern Asia, Australia and America in which counting always starts with the left little finger. In other regions it starts with the right little finger (see Map 2). In the middle part of Africa and in South America there are regions in which the right index finger is 1.

There are four connected regions in which counting starts with the thumb, namely one region in Western Europe, one in New Guinea and Melanesia, one in Middle and South America and one in Greenland. These regions are indicated in Map 2. It is an interesting fact that in all these regions we also find 20-count. Seidenberg writes[1]:

> These [distributions] may at first glance seem random, but it has been my observation that in almost every case in which the thumb is 1, the system is vigesimal.

Seidenberg uses this coincidence as an argument in favour of his hypothesis that every systematic counting system, such as 5–20-count, was invented only once. This seems to be a good probability argument. In fact, if the 5–20-system and thumb counting had been invented independently in (say) three or four regions, the coincidence between thumb counting and 20-count would be a highly improbable accident.



Two Roman counters showing the numbers VIII and VIIII formed by finger gestures (British Museum).

If we count on our fingers we usually *stretch* the fingers one after the other. The Japanese, on the other hand, *close* their fingers one after the other. Menninger[2] writes:

> A charming instance from the recent past is provided by an anecdote related by the Englishman R. Mason about the last world war (*... and the Wind Cannot Read*):
>
> Sabby was a Japanese girl in India, which was then at war with Japan. Her friend therefore introduced her as Chinese to an Englishman who had been living for a long time in India: 'Miss Wei'. 'Really'? He stretched his face forward and examined Sabby from close up, as if he were nearsighted. 'Nonsense', said he. 'Count with your fingers! Count to five!' Sabby looked shocked; she wasn't quite sure whether this extraordinary man was joking or mad. Hesitantly she raised her hand: 'One, two, three, four five,' she said uncertainly. Mr. Headley, burst out delightedly: 'There you are! Did you see that? Did you see how she did it? Began with her hand open and bent her fingers in one by one. Did you ever see a Chinese do such a thing? Never! The Chinese count like the English. Begin with the fist closed. She's Japanese!' he cried triumphantly.

As a final example we give the striking number words of the Dene-Dinje, a tribe of American Indians. These, from 1 to 5, are equivalent to:

1 'the end is bent' (little finger)
2 'it is bent once more' (ring finger)
3 'the middle is bent' (middle finger)
4 'only one remains' (thumb)
5 'my hand is finished'

Exercise 20

(a) Give the etymological relation often found (e.g. in Central Brazil) between number words for 5, 10 and 20 and parts of the body used for finger-and-toe counting.

1 *Op. cit.*, p. 263.
2 *Op. cit.*, p. 36.

(b) What evidence from finger counting does Seidenberg use to support his hypothesis that every systematic counting system was invented only once? Comment on the validity of this argument.

(c) Distinguish the finger counting method of Japan from that of China.

SA 20

(a) 5 related to 'hand'
 10 related to 'the hands'
 20 related to 'feet'

(b) The coincidence that, in almost every case observed, the counting system is vigesimal if the finger count takes the thumb as 1. The argument is reasonable because the coincidence between thumb counting and 20-count would be highly unlikely if the systems were independently invented in several different regions.

(c) The Japanese *close* their fingers one after the other; the Chinese *open* their fingers one after the other.

N1.11 BODY COUNTING

In several islands between New Guinea and Australia, and also in nearby New Guinea among the Papuans and in south-east Australia, there is a very curious counting method which may be called *body counting*. Seidenberg records that in Muralug Island and, with slight variations, in the other areas also the counting proceeds as follows:[1]

> Beginning with the little finger of the left hand, the natives counted up to 5 in the usual manner, and then, instead of passing to the other hand, or repeating the count on the same fingers, they expressed the numbers from 6 to 10 by touching and naming successively the left wrist, the left elbow, left shoulder, left breast, and sternum. The numbers from 11 to 19 were indicated by use, in inverse order, of the corresponding parts of the body on the right, the right little finger signifying 19—or 'nineteenth', rather, for it is fairly clear that the numbers are ordinal in character.

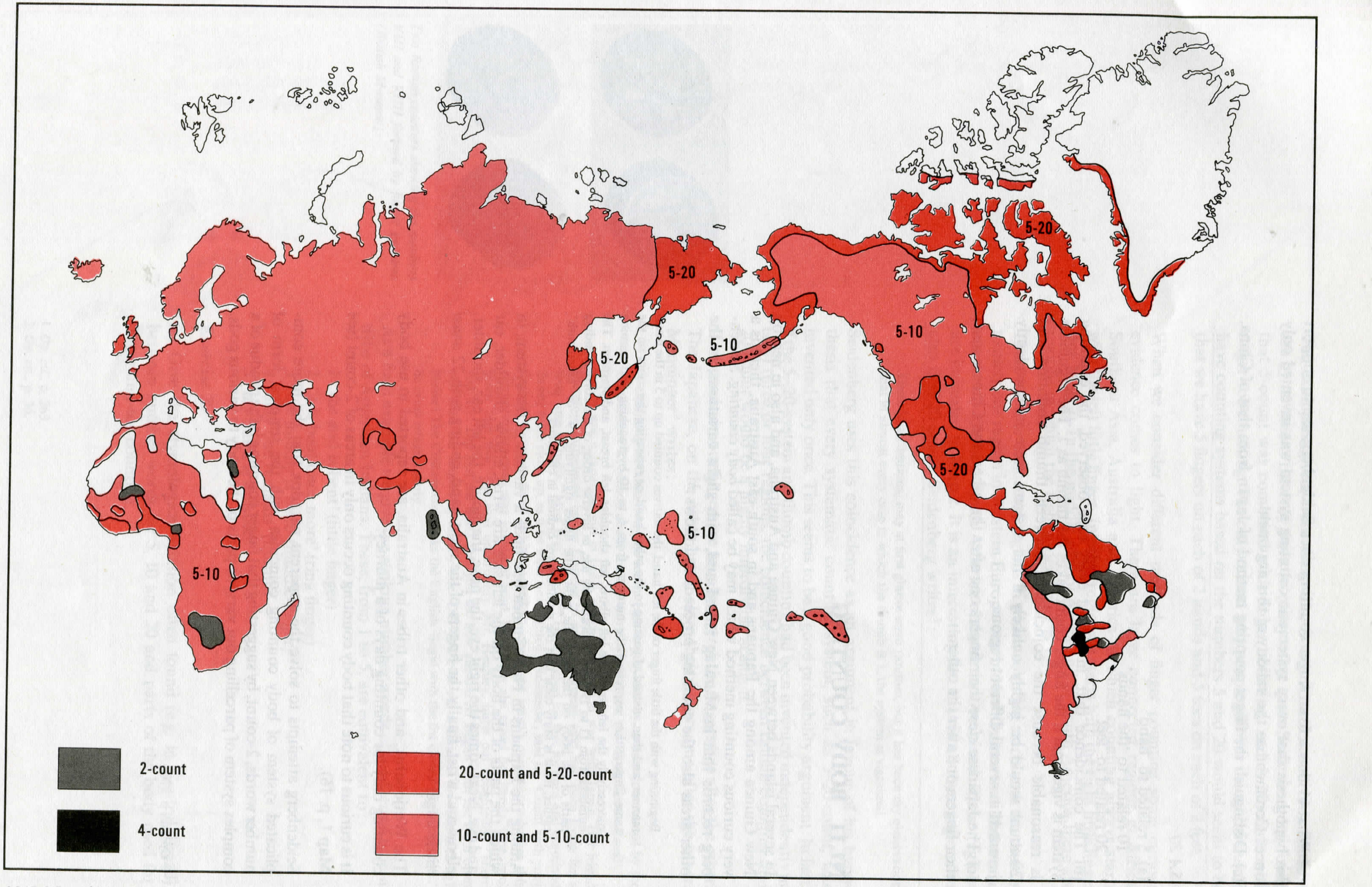
Among the Papuans in New Guinea one finds a variant of the system, in which the parts of the body on the left side are wrist, elbow, shoulder, ear and eye. Next comes the right eye, the nose, the mouth, the left ear, shoulder, elbow and wrist, finally the fingers of the right hand, ending with 22 *anusi* 'little finger'.

The Wotjobaluk, and other tribes in Australia, are reported to use body counting only for counting days of a journey.

It is curious to note that body counting occurs only in areas of 2-count (see Map 1, p. 16).

Seidenberg attempts to solve the apparent paradox that the more complicated system of body counting coincides with the simplest form of number words, 2-count, by suggesting that finger counting is the residue of a complex system of parcelling out various parts of the body to various gods.

1 *Op. cit.*, p. 270.



MAP 3 Distribution of counting systems.

N1.12 THE GEOGRAPHICAL DISTRIBUTION OF COUNTING-SYSTEMS

Map 3, p. 42, shows the geographical distribution of the various counting systems. It is a simplified version of Map XII in W. Schmidt's *Sprachfamilien und Sprachenkreise der Erde*, Heidelberg, 1926 (*Atlas*). Schmidt distinguishes ten counting methods from systemless counting to the pure decimal system (10-count). We have simplified the map by distinguishing only between

2-count
4-count
20- and 5–20-count
10- and 5–10-count

Schmidt also has small regions of 6-count and 40-count, and white regions about which he says nothing. We have left these regions white.

It is often difficult to say whether a counting-system is 5–10- or pure 10-, or whether it is 5–20- or pure 20-. The characteristic property of a 5–10- and 5–20-system is that 6, 7, 8 and 9 are pronounced as 5 + 1, 5 + 2, 5 + 3 and 5 + 4. In many cases, we find only traces of these decompositions. In those cases, where it is certain that the system is 5–10- or 5–20-, we have inserted the symbol '5–10' or '5–20'.

In Europe and Asia, the counting systems are mapped as they are today. In the other three continents only the counting methods of the aborigines are shown on the map and not the decimal system used by the white population.

Leaving the 4-system and the 5–10-system out of account, we are left with three principal systems, namely
(1) the pure 2-system
(2) the 20-system (5–20- and pure 20-)
(3) the 10-system.
The pure 2-system is very primitive: it cannot well be extended beyond 10. For this reason, it was superseded in most parts of the world by the better systems (2) and (3). The 2-system remained only in isolated parts of the Southern hemisphere.

The pure 10-system is the most efficient of all; it therefore tends to spread out over the earth. This tendency can be observed in the time of the Celts and Romans as well as in the last five centuries. In the middle ages, English was spoken only in Britain, now it is the principal language of, for example, North America and Australia. The Romance languages are now spoken in the whole of Southern Europe and South America and with these languages came the decimal system. The Chinese also have the decimal system and their civilization too spread out over a large area. So the spread of the decimal system is easy to explain. Today it covers by far the largest part of the surface of the earth.

The history of the 5–20-system is much more complicated and largely unknown. It was fully developed by the Aztecs and Mayas, in Mexico and Yucatan, and from this region it spread to the north and the south, replacing the less developed 5–10-count in the north and the primitive 2-count in the south. We find 5–20-count also among the Eskimos. They all speak dialects of one single language and their racial features point to an Asiatic origin. Hence, most probably, the 5–20-system was brought by the Eskimos from a centre in north-east Asia to the north coast of North America and to Greenland.

In Western Europe we have seen that the 20-system (perhaps originally a 5–20-system) once prevailed in a large area, including Spain, France, Great Britain, Ireland and some Scandinavian countries. When the Celtic and Germanic languages spread towards the West, the 20-system was largely replaced by the 10-system, but it maintained itself in the Basque language and traces of 20-count remained also in Irish, Scottish, French, Danish and other languages as we have seen.

All in all it seems that the 5–20-system has an intermediate position between the more primitive pure 2-system and the more efficient 10-system. In regions where the 10-system is fully developed up to 1000 or even beyond, it tends to spread out to replace the 5–20-system. On the other hand, in regions where the 5–20-system is fully developed (as in Middle America) this system is able to penetrate not only into regions of 2-count but even into regions of 10-count.

In a rather complicated argument, which we shall not consider here, Seidenberg suggests that the 5–10-system may be simply a cross between a quinariless 10-count and the 5–20-system. It is for this reason, that we have not included the 5–10-system amongst the three principal systems considered above, but have left it out of account along with the 4-system.

Exercise 21

(a) Name the three principal counting systems and comment on their geographical distribution.
(b) Which counting system is particularly associated with the Romance languages?
(c) In which systems are the number words for 6, 7, 8, 9 pronounced as $5 + 1, 5 + 2, 5 + 3, 5 + 4$ respectively?
(d) Give reasons for the spread of the decimal system over most of the world.
(e) In which European language has the 20-system maintained itself?
(f) Which American peoples are associated with the 5–20-system?
(g) Give a reason for regarding 5–10-count as of relatively less importance than the three principal counting systems.

SA 21

(a) 2-count, 20-count, 10-count.
2-count, since it is very primitive, has been superseded by better counting systems and exists only in isolated regions in Africa, Australia and South America. *20-count* is found in widely separated regions, but especially in Central America, amongst the Eskimos, in North-East Asia, in parts of Africa, and in the Basque country of Europe. *10-count* is almost universal except amongst isolated primitive peoples.
(b) The decimal system.
(c) 5–10- and 5–20-count.
(d) Its efficiency and its association with the spread of Indo-European and Chinese civilizations.
(e) The Basque language.
(f) Aztecs, Mayas and Eskimos.
(g) Because it is held to be a cross between pure 10-count and 5–20-count.

N1.13 THE ORIGIN OF COUNTING

We have not discussed the *origin of counting* as such in this unit. There are good reasons for this. The validity (or otherwise) of theories of the origin of counting is dependent to a considerable extent upon a satisfactory outcome of the *independent invention* versus *diffusion* controversy. Although we have presented considerable evidence which, in the main, can be held to support the diffusion hypothesis, we are not suggesting that the controversy is entirely resolved. (Indeed, we hope that you will digest the evidence presented and then come to conclusions of your own.)

The *independent inventionist* will look for the origin of counting in man's need to come to terms with and conquer his environment—that is, in practical applications of number. The *diffusionist*, on the other hand, seeks for the origin in a single centre of civilization, and he is therefore by no means restricted to a purely practical basis for the invention of counting.

We have not attempted to present the various *origin* arguments with their supporting evidence because the independent inventionists, for the most part, appear to take their hypothesis for granted and then seek for highly speculative psychological explanations of it. These explanations purport to determine what went on in the minds of our forbears many thousands of years ago and are not supported by any historical facts.[1]

Seidenberg has published a second paper which is directly concerned with the origin hypotheses. The argument concludes with the following sentence:[2]

> *Counting was invented in a civilised center. in elaboration of the Creation ritual, as a means of calling participants in ritual onto the ritual scene, once and only once, and then diffused.*

The arguments supporting this hypothesis are interesting but not conclusive (as Seidenberg himself admits) since much of the evidence considered is also compatible with a practical origin of counting. We refer those of you who are interested in a detailed study of these arguments to the paper itself.

Exercise 22

(a) Comment briefly on difficulties associated with discussion of the origin of counting.
(b) With which of the two hypotheses, *independent invention* and *diffusion*, is the hypothesis of a ritual origin of counting associated?

SA 22

(a) Discussion depends on a resolution of the *diffusion* versus *independent invention* controversy. Independent inventionists seek for speculative psychological explanations of the origin of counting.
(b) Diffusion.

---

1 Brief mention of these explanations is made in the associated radio programme *Diffusion of Counting*.
2 Seidenberg, A. (1962) The Ritual Origin of Counting, *Archive for History of Exact Sciences*, **2**, 1–40.

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